

A UNIFYING FORM FOR NOETHERIAN POLYNOMIAL REDUCTIONS

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ABSTRACT. Polynomial reduction is one of the main tools in computational algebra with innumerable applications in many areas, both pure and applied. Since many years both the theory and an efficient design of the related algorithm have been solidly established.

This paper presents a general definition of polynomial reduction structure, studies its features and highlights the aspects needed in order to grant and to efficiently test the main properties (noetherianity, confluence, ideal membership).

The most significant aspect of this analysis is a negative reappraisal of the role of the notion of term order which is usually considered a central and crucial tool in the theory. In fact, as it was already established in the computer science context in relation with termination of algorithms, most of the properties can be obtained simply considering a well-founded ordering, while the classical requirement that it be preserved by multiplication is irrelevant.

The last part of the paper shows how the polynomial basis concepts present in literature are interpreted in our language and their properties are consequences of the general results established in the first part of the paper.

1. INTRODUCTION

Polynomial reduction is one of the *cornerstones of Buchberger theory* [of] *remarkable generating sets for ideals in polynomial rings, [...] Gröbner bases, [...] an important branch in computer algebra* [3], the second cornerstone of Buchberger theory [9, 10, 13] being the notion of S-polynomials/S-pairs [11].

The crucial rôle of polynomial reduction in ideal theory can be stressed by remarking that it was one of the tools introduced by Gordan [30] for giving his proof of Hilbert's *Basissatz* [36] the other being the so-called Dickson Lemma [31].

In the meantime the same techniques were introduced by Riquier [56, 57] as a tool for solving partial differential equations and reformulated in Hilbert's language by Janet [38].

Actually the fruitful interaction with Hilbert's ideas granted to a group of researches in p.d.e. (and to Macaulay [43, 44]) to introduce, a generation before algebraists, notions and results related with Hilbert theory as generic initial ideals [18] [33], their connection with Borel/stable sets [58, 34] [24, 21] and their related Hironaka decomposition [39, 40] [37, 24], up to the current algorithm for computing resolutions [38] [60].

A very recent step in this linking new research with old results is [62] which, following a reformulation by Buchberger [12] of Gordan's approach, obtained a degree-bound evaluation for ideal membership test by merging Hermann's [35] and Dubé's [20] bounds.

Our aim is to reconsider polynomial reduction removing the irrelevant assumption that the set of the terms \mathcal{T} , which is the natural basis of the polynomial ring

$$\mathcal{P} = A[x_1, \dots, x_n] = A[\mathcal{T}],$$

is ordered by a term-ordering.

Such assumption, essentially, grants that in gaussian reduction once a *pivot* is fixed (*première membre* according Riquiet, *Anfangsglied* according Gordan, *leading term* according Buchberger) its rôle is preserved by monomial multiplication. If, however, following Riquiet and Janet and their reformulation in terms of *involution* [61], given a finite set \mathcal{F} of polynomials we allow to multiply each $f \in \mathcal{F}$ only by a restricted set of variables (*variables multiplicatives*) or, in general, by an order ideal τ_f of terms (*multiplier set*), it is sufficient to restrict the requirement of preserving leading terms to such subsets of multipliers.

This allows to consider well-orderings which are not semigroup ones but however grant a noetherian reduction. Naturally this does not contradicts Reeves-Sturmfels Theorem for the elementary reason that the theorem requires the application of the whole set of terms as multipliers.

Of course, in order that a noetherian polynomial reduction could grant ideal membership test, it must satisfy confluency (or, at least, local confluency) or, more simply, all polynomials $\sigma f, f \in \mathcal{F}, \sigma \in \mathcal{T} \setminus \tau_f$, must be reducible to zero.

Our aim of considering termordering-free reductions and thus dealing with potentially infinite confluence tests is due to significant applications.

The main one is the study of Hilbert scheme; it is well-known [4] that deformations of the Gröbner basis of an ideal \mathfrak{l} in the polynomial ring \mathcal{P} are a flat family and can thus be applied for studying geometrical deformations of the scheme \mathcal{X} defined by \mathfrak{l} . However such families of deformations in general cover only locally closed subschemes of Hilbert scheme and are not sufficient to study neighborhoods of deformations of \mathcal{X} , *id est* opens of Hilbert scheme. Such opens can be instead obtained by considering [7] those ideals \mathfrak{l}' of \mathcal{P} which share with \mathfrak{l} a fixed monomial basis of the quotient \mathcal{P}/\mathfrak{l} . In order to determine the family of all such ideals \mathfrak{l}' of \mathcal{P} it is thus necessary to move out from the Groebnerian enclosure and to generalize such techniques and results weakening the assumptions, mainly the use of termorderings.

Termordering-free bases of polynomial ideals worked out at this aim have been developed in [17] for the homogeneous case and in [5] for the affine one. The construction of the flat family parameterizing all ideals \mathfrak{l} with a fixed monomial set as a basis of the quotient ring \mathcal{P}/\mathfrak{l} , obtained as a scheme representing a functor, has been discussed in [42].

Another potential application is semigroup rings; it is well-known that an elementary adaptation of Buchberger Theory to semigroup rings is impossible exactly because for a polynomial f it is possible that in the set $\{\sigma f, \sigma \text{ a term}\}$ even all terms in the support of f could become leading term; for the restricted case of group rings there is an elegant solution [59] which easily produces the (at most two) elements of the Gröbner basis; the general solution [45, 46] is able to reduce the test to an iterative computation of S-polynomials among the given generators in the semigroup ring and the relations defining the semigroup as a quotient of the free semigroup; a recent investigation [51, IV.47.9.2, IV.50.13.5] indicates that the notion of multiplier set arises naturally in that

setting and suggests that an adaptation of our notion to the non-commutative case could be interesting.

We define a *reduction structure* (Definition 3.1) as the assignment of

- a finite set $M \subset \mathcal{T}$ of terms generating, not uniquely, a semigroup ideal $J \subset \mathcal{T}$, and to each element $x^\alpha \in M$
- a *multiplier set* $\tau_\alpha \subset \mathcal{T}$ (and the related *cone* $\text{cone}(x^\alpha) := \{x^{\alpha+\eta} | x^\eta \in \tau_\alpha\}$) such that

$$(1) \quad J = \bigcup_{x^\alpha \in M} \text{cone}(x^\alpha),$$

- and a finite subset $\lambda_\alpha \subset \mathcal{T} \setminus \tau_\alpha$.

We then consider a related *marked set* of polynomials

$$\mathcal{F} := \left\{ f_\alpha := x^\alpha + \sum_{x^\gamma \in \lambda_\alpha} c(f_\alpha, x^\gamma) x^\gamma \right\}$$

and define a Buchberger-Gordan reduction in terms of \mathcal{F} setting

$$g := \sum_{x^\epsilon \in \mathcal{T}} c(g, x^\epsilon) x^\epsilon \rightarrow h := g - c(g, x^{\alpha+\eta}) x^\eta f_\alpha \iff c(g, x^{\alpha+\eta}) \neq 0, x^\eta \in \tau_\alpha.$$

As we have remarked above, the choice of restricting the set of *reducers* to the subsets $\{x^\eta f_\alpha | x^\eta \in \tau_\alpha\}$ sufficient to grant (1) and thus fixing *pivots* grants a unique reduction but noetherianity is obtained only if each *Anfangsglied* is preserved in this restricted set; we obtain so by following the tools used for proving termination of programs and term rewriting systems [19, 47] and

fixing

- a well founded ordered set $(E, >)$ and
- a map $\varphi: \mathcal{T} \rightarrow E$ s.t.

$$\varphi(x^{\alpha+\eta}) > \varphi(x^{\gamma+\eta}) \text{ for each } x^\alpha \in M, x^\gamma \in \lambda_\alpha, x^\eta \in \tau_\alpha.$$

Once noetherianity is granted, if the order ideal $N(J) := \mathcal{T} \setminus J$ is a free basis as A -module of the quotient of the polynomial ring modulo the ideal I generated by the marked set \mathcal{F} (which in this case would be labeled *marked basis*),

$$\mathcal{P}/I \cong A[N(J)],$$

and the related polynomial reduction is confluent, then, for each polynomial g the unique element $l \in A[N(J)]$ such that $g \rightarrow_\star l$ is its canonical form.

While Buchberger Theory of Gröbner bases is the “natural” model for polynomial reduction, we considered also some recent and promising alternative to Buchberger Algorithm and used them to formulize different term-ordering free polynomial reductions:

- Gebauer–Möller Staggered Linear Bases [25] (completely superseded by F_5 [23, 2, 22]) suggested us to restrict a reduction structure to a substructure with disjoint cones, *id est*

$$\text{cone}(x^\alpha) \cap \text{cone}(x^{\alpha'}) = \emptyset, \forall x^\alpha, x^{\alpha'} \in M, x^\alpha \neq x^{\alpha'};$$

- the original *completion* construction and related notion of *multiplicative variables* introduced by Janet [38] and which is behind *involutive bases* [63, 26, 27] suggested us the notion of *stably ordered reduction structures* thanks of a careful analysis of the needed properties of the map $\varphi: \mathcal{T} \rightarrow E$.

Fixed the notation (Section 2) and introduced the definition and related notions of *reduction structure* (Section 3), we discuss (Section 4) *marked sets* and the associated rewriting rule \rightarrow , focusing on its main properties (Section 5), noetherianity, weak noetherianity and their relation with the orderedness of the related reduction structure (Theorem 5.7) and with Reeves-Sturmfels Theorem (Theorem 5.9), the structure of the related Gröbner representation (Section 6, Proposition 6.2), confluency (Section 7), canonical forms (Section 8) and the functorial description of reduction structures (Section 9).

Next we discuss both structures with disjoint cones (Sections 10-11) and stably ordered reduction structures (Section 12). Finally (Sections 13-14) we cover the most important types of known polynomial bases consistent with a term order reformulating them in our language: Gröbner bases [9, 10, 13] (Section 13.1), Gebauer–Möller Staggered Linear Bases [25] (Section 13.2), Janet’s completion [38] (Section 13.3), Janet-like bases [28, 29] (Section 13.4), Janet-Pommaret *involutive* construction [39, 40, 53, 26, 27] (Section 14.1), Robbiano’s notion of border basis [41] (Section 14.2).

2. NOTATION.

Consider the polynomial ring

$$\mathcal{P} := A[x_1, \dots, x_n] = \bigoplus_{d \in \mathbb{N}} \mathcal{P}_d$$

in n variables and coefficients in the base ring A .

When an order on the variables comes into play, we consider $x_1 < x_2 < \dots < x_n$.

The set of terms in the variables x_1, \dots, x_n is

$$\mathcal{T} := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}.$$

Given a term $x^\alpha \in \mathcal{T}$, we set

$$\max(x^\alpha) = \max\{x_i : x_i \mid x^\alpha\}, \min(x^\alpha) = \min\{x_i : x_i \mid x^\alpha\}$$

the maximal and the minimal variable appearing in x^α with nonzero exponent.

If $\{x_{j_1}, \dots, x_{j_r}\} \subset \{x_1, \dots, x_n\}$, we define

$$\mathcal{T}[x_{j_1}, \dots, x_{j_r}] := \{x_{j_1}^{\alpha_{j_1}} \cdots x_{j_r}^{\alpha_{j_r}}, (\alpha_{j_1}, \dots, \alpha_{j_r}) \in \mathbb{N}^r\}.$$

For every polynomial $f \in \mathcal{P}$, $\deg(f)$ is its usual degree and $\deg_i(f)$ is its degree with respect to the variable x_i . For each $p \in \mathbb{N}$, and for all $W \subseteq \mathcal{P}$,

$$W_p := \{f \in W : f \text{ homogeneous and } \deg(f) = p\};$$

in particular:

$$\mathcal{T}_p := \{x^\alpha \in \mathcal{T} : \deg(x^\alpha) = p\}.$$

Once a well-ordering $<$ is fixed in \mathcal{T} then each $f \in \mathcal{P}$ has a unique representation as an ordered linear combination of terms $t \in \mathcal{T}$ with coefficients in A :

$$f = \sum_{i=1}^s c(f, t_i) t_i : c(f, t_i) \in A \setminus \{0\}, t_i \in \mathcal{T}, t_1 > \dots > t_s.$$

The *support* of f is the set $\text{Supp}(f) := \{t : c(f, t) \neq 0\}$; we further denote $\mathbf{T}(f) := t_1$ the *maximal term* of f , $\text{lc}(f) := c(f, t_1)$ its *leading coefficient* and $\mathbf{M}(f) := c(f, t_1)t_1$ its *maximal monomial*.

Definition 2.1 (Buchberger). [9, 10][51, II, Definition 25.1.2.]

For each $f, g \in \mathcal{P}$ such that $\text{lc}(f) = 1 = \text{lc}(g)$, the polynomial

$$S(g, f) := \frac{\text{lcm}(\mathbf{T}(f), \mathbf{T}(g))}{\mathbf{T}(f)} f - \frac{\text{lcm}(\mathbf{T}(f), \mathbf{T}(g))}{\mathbf{T}(g)} g$$

is called the *S-polynomial* of f and g .

Definition 2.2. Let $F = \{f_1, \dots, f_s\}$ be an ordered set of polynomials. The module

$$\text{Syz}(F) = \{(g_1, \dots, g_s) \in \mathcal{P}^s, \sum_{i=1}^s g_i f_i = 0\}$$

is the *syzygy module* of F and any element (g_1, \dots, g_s) in $\text{Syz}(F)$ is called a *syzygy* of F .

Definition 2.3. A set $\mathbf{N} \subset \mathcal{T}$ is called *order ideal* if

$$\forall x^\alpha \in \mathcal{T}, x^\beta \in \mathbf{N} : x^\alpha | x^\beta \Rightarrow x^\alpha \in \mathbf{N}.$$

Observe that \mathbf{N} is an order ideal if and only if the complementary set $I := \mathcal{T} \setminus \mathbf{N}$ is a *semigroup ideal*, i.e. $\forall x^\eta \in \mathcal{T}, x^\gamma \in I \Rightarrow x^{\eta+\gamma} \in I$.

If I is either a monomial ideal or a semigroup ideal, we will denote by $\mathbf{N}(I)$ the order ideal $\mathbf{N} := \mathcal{T} \setminus I$ and by $\mathbf{G}(I)$ its *monomial basis*, namely the minimal set of terms generating I .

Definition 2.4 ([55]). A *marked polynomial* is a polynomial $f \in \mathcal{P}$ together with a fixed term $Ht(f)$ that appears in f with coefficient 1_A .

3. INTRODUCING REDUCTION STRUCTURES

Definition 3.1. A *reduction structure* (RS for short) \mathcal{J} in \mathcal{T} is a 3-tuple

$$(M, \lambda := \{\lambda_\alpha, x^\alpha \in M\}, \tau := \{\tau_\alpha, x^\alpha \in M\})$$

that satisfies the following conditions

- M is a *finite* set of terms; we will denote by J the semigroup ideal generated by M ;
- $\tau_\alpha \subseteq \mathcal{T}$ is an order ideal, called *multiplicative set* of x^α , s.t. $\bigcup_{x^\alpha \in M} \text{cone}(x^\alpha) = J$, where $\text{cone}(x^\alpha) := \{x^{\alpha+\eta} \mid x^\eta \in \tau_\alpha\}$ is the *cone* of x^α ;
- λ_α is a subset of $\mathcal{T} \setminus \text{cone}(x^\alpha)$ that we call *tail set* of x^α .

Lemma 3.2. Let \mathcal{J} be a reduction structure. Then, there is at least a term $x^\alpha \in M$ s.t. $\tau_\alpha = \mathcal{T}$. In particular it holds $\mathcal{T} = \bigcup_{x^\alpha \in M} \tau_\alpha$.

Proof. Suppose that the assertion is false and, for each $x^{\alpha_i} \in M$, choose a term x^{η_i} not belonging to τ_{α_i} . We denote x^β the product of the terms $x^{\alpha_i+\eta_i}$, $x^{\alpha_i} \in M$. By definition of reduction structure there is a term in M , let it be x^{α_1} , whose cone contains x^β , so $x^{\beta-\alpha_1}$ is a multiple of x^{η_1} and belongs to τ_{α_1} . Since τ_{α_1} is an order ideal, it contains also x^{η_1} , leading to a contradiction. \square

Definition 3.3. We will call *substructure* of $\mathcal{J} = (M, \lambda, \tau)$ each RS of the form $\mathcal{J}' = (M, \lambda, \tau')$ s.t. for each $x^\alpha \in M$ it holds $\tau'_\alpha \subseteq \tau_\alpha$. In this case we will write $\mathcal{J}' \subseteq \mathcal{J}$.

Reduction Structures of the following type will be important in the whole paper

Definition 3.4. A Reduction Structure \mathcal{J} is:

- *homogeneous* if $\forall x^\alpha \in M$ it holds $\lambda_\alpha \subset \mathcal{T}_{|\alpha|}$;
- with *finite tails* if $\forall x^\alpha \in M$ it holds $|\lambda_\alpha| < \infty$
- with *reduced tails* if $\forall x^\alpha \in M$ it holds $\lambda_\alpha \subseteq \mathbf{N}(J)$
- *consistent with a term order* \prec if $\forall x^\alpha \in M$ and $\forall x^\gamma \in \lambda_\alpha$ it holds $x^\alpha \succ x^\gamma$
- with *maximal cones* if $\forall x^\alpha \in M$ it holds $\tau_\alpha = \mathcal{T}$;
- with *disjoint cones* if $\forall x^\alpha, x^{\alpha'} \in M, x^\alpha \neq x^{\alpha'}$, it holds $\text{cone}(x^\alpha) \cap \text{cone}(x^{\alpha'}) = \emptyset$;
- with *multiplicative variables* if $\forall x^\alpha \in M$ it holds

$$\tau_\alpha = \mathcal{T}[\mu_\alpha], \text{ where } \mu_\alpha \subseteq \{x_1, \dots, x_n\}.$$

More generally, we will call x_i *multiplicative variable* for $x^\alpha \in M$ if $x_i \tau_\alpha \subset \tau_\alpha$.

As we will see in details in Section 13, there are reduction structures that give the natural framework in which we find Gröbner bases and their properties. They are built as follows: M is any finite set of terms; for each term $x^\alpha \in M$, τ_α is the whole \mathcal{T} and λ_α is the sets of terms lower than x^α w.r.t. a fixed term order. In the terminology just introduced, these reduction structures are consistent with a term order, have multiplicative variables and maximal cones.

On the other hand, our definition also include strange reduction structures that cannot be included neither in a standard Gröbner framework nor in any other type of polynomial bases that (in our knowledge) are already present in literature.

Example 3.5. In $A[x, y]$ let us consider the reduction structure \mathcal{J} given by

- $M = \{x^3, xy, y^3\}$;
- $\lambda_{x^3} = \lambda_{y^3} = \{x^2y, xy^2, x^2, xy, y^2, x, y, 1\}$, $\lambda_{xy} = \{x, y, 1\}$
- $\tau_{x^3} = \mathcal{T}[x]$, $\tau_{xy} = \mathcal{T}[x, y]$, $\tau_{y^3} = \mathcal{T}[y]$.

This reduction structure is not coherent with a term order; however it has two of the most useful features that we can expect by a polynomial rewriting rule and that we will discuss in the following sections: noetherianity and confluence.

4. MARKED SETS AND REWRITING RULES

We use Reduction Structures in order to investigate when and how marked polynomials can be efficiently applied as as rewriting rules (for theoretical results on polynomial rewriting rules see [19, 14, 8]).

Definition 4.1. Given a RS $\mathcal{J} = (M, \lambda, \tau)$, consider for each $x^\alpha \in M$ a monic marked polynomial $f_\alpha \in \mathcal{P}$ s.t. $\text{Ht}(f_\alpha) = x^\alpha$ and $\text{Supp}(f_\alpha - x^\alpha) \subset \lambda_\alpha$. We will call *head* of f_α the term x^α and *tail* of f_α the difference $f_\alpha - x^\alpha$.

The set $\mathcal{F} = \{f_\alpha\}_{x^\alpha \in M}$ of polynomials in \mathcal{P} will be called *marked set* on \mathcal{J} .

We will denote by $\tau\mathcal{F}$ the set of the $x^\eta f_\alpha$ s.t. $x^\eta \in \tau_\alpha$, by $\langle \tau\mathcal{F} \rangle$ the A -module generated by these elements and by (\mathcal{F}) the ideal of \mathcal{P} generated by \mathcal{F} .

We can associate to a marked set \mathcal{F} on \mathcal{J} a reduction procedure $\xrightarrow{\mathcal{F}\mathcal{J}}$.

A polynomial $g \in \mathcal{P}$ whose support is contained in $N(J)$ is *reduced* w.r.t. J or is a *J-remainder*. If g is not reduced w.r.t. J , the base step of reduction applied to g consists on finding a term $x^\gamma \in \text{Supp}(g)$ belonging also to J and choosing an element $x^\alpha \in M$ s.t. $x^\gamma = x^{\alpha+\eta} \in \text{cone}(x^\alpha)$. If $c = c(g, x^\gamma) \in A$ is the coefficient of x^γ in g , we set $h := g - cx^\eta f_\alpha$. We define the reduction process as the transitive closure of the base steps; we will write $g \xrightarrow{\mathcal{F}\mathcal{J}} l$ to say that we can get l applying a sequence of base steps of reduction starting from g .

Notice that there could be several terms in $\text{Supp}(g) \cap J$ and that each of them can belong to several cones. Therefore, the reduction procedure of a general polynomial g is in general far from being unique.

Definition 4.2. A *rewriting rule* is a couple $(\mathcal{F}, \xrightarrow{\mathcal{F}\mathcal{J}})$, where \mathcal{F} is a marked set over a RS \mathcal{J} and $\xrightarrow{\mathcal{F}\mathcal{J}}$ is the procedure constructed above.

Remark 4.3. It is clear from the definition that the rewriting rule does not depend only on the marked set \mathcal{F} , but also on the RS to which \mathcal{F} is associated. Indeed a same set of marked polynomials can be related to several RSs giving rise to essentially different reduction procedures.

For instance, the marked sets on \mathcal{J} and on a substructure \mathcal{J}' are exactly the same, but in general the reduction processes associated to the same \mathcal{F} are different. More precisely, we have

$$g \xrightarrow{\mathcal{F}\mathcal{J}'} h \implies g \xrightarrow{\mathcal{F}\mathcal{J}} h.$$

If \mathcal{F} is a marked set over a RS $\mathcal{J} = (M, \lambda, \tau)$ we can also consider \mathcal{F} as marked over $\mathcal{J}' := (M, \{\lambda_\alpha \cap \text{Supp}(f_\alpha)\}, \tau)$, getting the same rewriting rule. The terms of each λ_α not appearing in f_α are irrelevant in the reduction steps involving f_α . Moreover, they are irrelevant for the steps not involving f_α . This yields an advantage: we can work over reduction structures with finite tails. Anyway, notice that the set of marked sets over \mathcal{J}' is a proper subset of the analogous over \mathcal{J} .

In what follows, each RS will have finite tails.

5. NOETHERIANITY I: WELL FOUNDED ORDERINGS

In this section we discuss some relations between the different types of reduction structures we have introduced in relation with the noetherianity of the rewriting rules.

Definition 5.1. If $g \xrightarrow{\mathcal{F}\mathcal{J}} l$ and $\text{Supp}(l) \subset N(J)$, we will write $g \xrightarrow{\mathcal{F}\mathcal{J}}_* l$ and call l a *reduced form* (or *J-remainder*) of g , obtained via \mathcal{F} . If such a polynomial l exists, we will say that g has a *complete reduction w.r.t. \mathcal{F}* .

We will say that $\xrightarrow{\mathcal{F}\mathcal{J}}$ is *noetherian* if all the possible sequences of reduction on every polynomial g stop at reduced forms after a finite number of base steps.

We will call \mathcal{J} *noetherian* if for each marked set \mathcal{F} on \mathcal{J} , the rewriting rule $\xrightarrow{\mathcal{F}\mathcal{J}}$ is noetherian. The reduction structure \mathcal{J} is *weakly noetherian* if it has a noetherian substructure.

If \mathcal{J} is weakly noetherian, each polynomial g has a complete reduction $g \xrightarrow{\mathcal{F}\mathcal{J}}_* l$, though there could be also infinite sequences of base steps of reduction starting on g .

Lemma 5.2. *Let \mathcal{J} be a RS. Then*

- (i) *if \mathcal{J} is noetherian, then it is also weakly noetherian;*
- (ii) *if \mathcal{J} has disjoint cones, then also the converse holds true;*
- (iii) *if $\mathcal{J}' \subseteq \mathcal{J}$ and \mathcal{J} is noetherian, then also \mathcal{J}' is noetherian;*
- (iv) *if $\mathcal{J}' \subseteq \mathcal{J}$ and \mathcal{J}' is weakly noetherian, then \mathcal{J} is weakly noetherian.*

Proof. All these properties are trivial consequences of the definitions. We only observe for (ii) that a reduction structure with disjoint cones has no proper substructures. \square

In order to find some effective way to check the noetherianity of a reduction structure, we now exploit arguments and results concerning the termination of algorithms based on rewriting rules, that have been developed mainly in the computer science context. They state a closed relation between the noetherianity and the presence of a suitable well founded ordering.

We recall that an ordering $<$ on a set W is called *well founded* if each nonempty subset of W contains minimal elements.

Definition 5.3. We say that a reduction structure \mathcal{J} is *ordered*, if there is a well founded ordered set $(W, >)$ and an *ordering function* $\varphi: \mathcal{T} \rightarrow W$ s.t.

$$\forall x^\alpha \in M, x^\gamma \in \lambda_\alpha, x^\eta \in \tau_\alpha \text{ it holds } \varphi(x^{\alpha+\eta}) > \varphi(x^{\gamma+\eta}).$$

All the reduction structures coherent with a term order \prec are obviously ordered taking $(W, <) = (\mathcal{T}, \prec)$ and $\varphi = Id_{\mathcal{T}}$. However there are ordered reduction structures that are not coherent with a term order.

Example 5.4. Let us consider Example 5.14 of [16], namely the reduction structure \mathcal{J} given by

- $M := \{x^3, xy, xy^2, y^3\};$
- $\tau_{x^3} = \tau_{xy} = \tau_{xy^2} := \mathcal{T}[x], \tau_{y^3} := \mathcal{T}[x, y];$
- $\lambda_{x^3} := \lambda_{xy^2} := \lambda_{y^3} := \emptyset, \lambda_{xy} := \{x^2, y^2\}.$

We can see that \mathcal{J} is ordered considering $W = \mathcal{T}$, $\varphi = Id_{\mathcal{T}}$ and a well founded order on \mathcal{T} . We list here all total degree-compatible well-founded orders $<$ which satisfy the formulation given in [51, IV.Rem.56.11.6] for describing the result of [17]; of course *this list does not contain any semigroup ordering*.

Namely¹

$$1 < \{x, y\} < \{x^2, y^2\} < xy$$

Next let us fix any of the six orderings of the set $\{x^3, xy^2, y^3\}$ and denote it $\alpha < \beta < \gamma$. The sequence then continues increasingly ordering the terms of degree $n, n = 3, 4, 5, \dots$; at each value n we have

$$x^{n-1}y < \alpha x^{n-3} < \beta x^{n-3} < \gamma x^{n-3} < x^{n-4}y^4 < \dots < x^{n-i}y^i < x^{n-i-1}y^{i+1} < \dots < y^n$$

¹the notation $A < B$ for two sets A, B means that all elements of A must be less than each element of B , but in each set the elements can be freely ordered. For instance $\{a, b\} < \{c, d\}$ characterize each of the four orderings

$$a < b < c < d, b < a < c < d, a < b < d < c, b < a < d < c$$

Example 5.5. The reduction structure of Example 3.5 is ordered by the non-injective function $\varphi: \mathcal{T} \rightarrow (\mathbb{N}, <)$ given by $\varphi(x^m) = \varphi(y^m) = 2m$ and $\varphi(x^{r+1}y^{s+1}) = 2r + 2s + 3$, for every $m, r, s \in \mathbb{N}$.

In order to connect this definition of ordered reduction structure to the rewriting rules on it, we first extend the ordering function $\varphi: \mathcal{T} \rightarrow W$ to a function from polynomials to the set $\mathcal{M}(W)$ of finite multisets with elements taken from the set W .

For any given function $\varphi: \mathcal{T} \rightarrow W$, let $\bar{\varphi}: \mathcal{P} \rightarrow \mathcal{M}(W)$ the function that associate to a polynomial $f \in \mathcal{P}$ the multiset $\varphi(\text{Supp}(f))$, where an element $e \in W$ appears with multiplicity given by the number of terms $x^\alpha \in \text{Supp}(f)$ such that $\phi(x^\alpha) = e$.

We can interpret $\bar{\varphi}$ as an ordering function endowing $\mathcal{M}(W)$ by the partial order \ll introduced in [19]: roughly speaking, $B \ll A$ if B is obtained from A by replacing some occurrence of its elements by lower elements. More precisely, giving to \setminus and \cup the usual meaning they have when multisets are involved:

$$B \ll A \iff B = (A \setminus Y) \cup Z$$

where $Y \subset A$ and $\forall z \in Z \exists y \in Y$ such that $z < y$.

Theorem 5.6. [Dershowitz-Manna [19]] $(W, <)$ is well founded if and only if $(\mathcal{M}(W), \ll)$ is.

It is quite obvious that for every marked set \mathcal{F} on the reduction structure \mathcal{J} ordered by φ , $f \xrightarrow{\mathcal{F}\mathcal{J}} g$ implies $\bar{\varphi}(f) \gg \bar{\varphi}(g)$. We can then reformulate in our framework a well know results by Z. Manna and S. Ness concerning the termination of programs ([47],[19]).

Theorem 5.7. Let \mathcal{J} be a reduction structure. Then

$$\mathcal{J} \text{ is ordered} \iff \mathcal{J} \text{ is noetherian.}$$

Remark 5.8. If $(W, <)$ is well founded, then a function $\varphi: \mathcal{T} \rightarrow W$ induces a well founded order on \mathcal{T} , given by $x^\alpha > x^\beta$ if and only if $\varphi(x^\alpha) > \varphi(x^\beta)$. Analogously, by Theorem 5.6, we also obtain a well founded order on \mathcal{P} via $\bar{\varphi}$. Therefore, without losing in generality we may assume that the ordering function of an ordered reduction structure is the identity, namely that $<$ is a well founded order on \mathcal{T} and \ll is a well founded order on \mathcal{P} .

Due to the above result, **in the following we consider noetherian and ordered as synonyms for what concerns the reduction structures.** Therefore, to every noetherian RS we associate an ordering function φ from \mathcal{T} to a suitable well founded set $(W, <)$ and its extension $\bar{\varphi}$ from \mathcal{P} to the well founded set of finite multisets $(\mathcal{M}(W), \ll)$.

We conclude this section with a reformulation of a well known result by Reeves and Sturmfels in our language.

Theorem 5.9 (Reeves-Sturmfels, [55]). Let $\mathcal{J} = (M, \lambda, \tau)$ be a SR with maximal cones. Then

$$\mathcal{J} \text{ is noetherian} \iff \mathcal{J} \text{ is consistent with a term-order.}$$

Proof. Each SR which is consistent with a term order \prec is ordered (so it is noetherian); it is enough to take $(E, >) = (\mathcal{T}, \succ)$ and $\varphi = Id_{\mathcal{T}}$.

To prove the converse, we use the same reasoning of Reeves-Sturmfels. Consider the subset $D := \{\alpha - \gamma \mid x^\alpha \in M, x^\gamma \in \lambda_\alpha\} \subset \mathbb{Z}^n$. If there is a vector $\mathbf{w} \in \mathbb{Z}^n$ with positive entries s.t. $(\alpha - \gamma) \cdot \mathbf{w} > 0$ for each $\alpha - \gamma \in D$, then \mathcal{J} is consistent with the term order associated to an arbitrary matrix with integer entries s.t. \mathbf{w} is its first row.

If there is no such a vector, by convexity reasons, the zero vector can be obtained as a linear combination of vectors in D with positive integer coefficients, i.e. there are r terms $x^{\alpha_1}, \dots, x^{\alpha_r} \in M$ (possibly non distinct) and elements $x^{\gamma_i} \in \lambda_{\alpha_i}$ s.t. $x^{\alpha_1} \dots x^{\alpha_r} = x^{\gamma_1} \dots x^{\gamma_r}$.

Construct the marked set $\{f_\alpha = x^\alpha - \sum_{x^\gamma \in \lambda_\alpha} C_{\alpha,\gamma} x^\gamma, x^\alpha \in M\}$ in the polynomial ring with coefficient in $A = \mathbb{Q}[C]$, where C is the set of variables $C_{\alpha,\gamma}$, all distinct, one for each $x^\alpha \in M$ and $x^\gamma \in \lambda_\alpha$. We remind now that we are considering only the case of RSs with finite tails.

Then we can reduce the polynomial $g_0 := x^{\alpha_1} \dots x^{\alpha_r}$ w.r.t. f_{α_1} , since $\tau_{\alpha_1} = \mathcal{T}$. This way, we get a polynomial g_1 whose support contains $x^{\gamma_1} \cdot x^{\alpha_2} \dots x^{\alpha_r}$. We apply now to this term a second base step w.r.t. f_{α_2} , getting a polynomial whose support contains $x^{\gamma_1} \cdot x^{\gamma_2} \cdot x^{\alpha_3} \dots x^{\alpha_r}$. We proceed this way, reducing the term $x^{\gamma_1} \dots x^{\gamma_{i-1}} \cdot x^{\alpha_i} \dots x^{\alpha_r}$ at the i -th step.

Note that actually this term appears, since the coefficients of the terms in g_{i-1} have degree $i - 1$ in the variables C , whereas in g_i the terms in $\text{Supp}(f_{\alpha_i} - x^{\alpha_i})$ all have a coefficient in degree i . \square

The first example of noetherian reduction structures one can think about are given by Groebner bases. The theorem above states that they are all of them if we require both noetherianity and maximal cones. In the following example we present a reduction structure which is *not* consistent with a term-ordering while noetherian.

By any similar reduction structure, we can obtain examples of weakly noetherian reduction structures with maximal cones, though non-consistent with a term order. Indeed, if $\mathcal{J}' = (M', \lambda', \tau')$ is a noetherian, then $\mathcal{J} = (M = M', \lambda = \lambda', \{\tau_\alpha = \mathcal{T}\})$, of which \mathcal{J}' is a substructure, is weakly noetherian and has maximal cones.

Example 5.10. In $A[x, y]$ we consider

- $M = \{xy, x^3, y^3, xy^2, x^2y^2\}$;
- $\tau_{xy} = \tau_{x^3} = \mathcal{T}[x]$, $\tau_{y^3} = \tau_{xy^2} = \mathcal{T}[y]$, $\tau_{x^2y^2} = \mathcal{T}[x, y]$;
- $\lambda_{xy} = \{x^2, y^2\}$, $\lambda_{x^3} = \lambda_{y^3} = \lambda_{xy^2} = \lambda_{x^2y^2} = \emptyset$.

The reduction structure $\mathcal{J} = (M, \lambda, \tau)$ is trivially *non* consistent with a term-ordering but is noetherian.

In fact:

- if $v \in \text{cone}(x^3) \cup \text{cone}(y^3) \cup \text{cone}(xy^2) \cup \text{cone}(x^2y^2)$, trivially $v \rightarrow 0$;
- $xy \rightarrow x^2 + y^2 \in \langle \mathbf{N}(J) \rangle$
- $x^2y = x(xy) \rightarrow x(x^2 + y^2) = x^3 + xy^2 \rightarrow 0$
- $x^{i+3}y = x^{i+2}(xy) \rightarrow x^{i+2}(x^2 + y^2) = x^{i+1} \cdot x^3 + y \cdot x^i \cdot x^2y^2 \rightarrow 0, i \geq 0$.

Note that the example does not contradict Reeves-Sturmfels Theorems for the simple reason that the cones are *not* maximal.

6. NOETHERIANITY II: LOWER REPRESENTATIONS OF POLYNOMIALS

In this section we relate the reduction of a polynomial g and the operation of subtracting from g a polynomial in $\tau\mathcal{F}$. We recall that for a given marked set \mathcal{F} over a RS

$\mathcal{J} = (M, \lambda, \tau)$, we denote by $\tau\mathcal{F}$ the set of polynomials $x^\gamma f_\alpha$ with $f_\alpha \in \mathcal{F}$ and $x^\gamma \in \tau_\alpha$, and by $\langle \tau\mathcal{F} \rangle$ the A -module generated by $\tau\mathcal{F}$.

Definition 6.1. Let \mathcal{F} be a marked set over a RS \mathcal{J} and let g be any polynomial in $\langle \tau\mathcal{F} \rangle$.

If $g = \sum_{i=1}^r c_i x^{\eta_i} f_{\alpha_i}$ with $c_i \in A$ and $x^{\eta_i} f_{\alpha_i}$ distinct elements of $\tau\mathcal{F}$, we say that the writing $\sum_{i=1}^r c_i x^{\eta_i} f_{\alpha_i}$ is a *representation of g by $\tau\mathcal{F}$* .

If, moreover, \mathcal{J} is noetherian with ordering function φ and x^δ is any term, we say that a representation $g = \sum_{i=1}^r c_i x^{\eta_i} f_{\alpha_i}$ by $\tau\mathcal{F}$ is a x^δ -*lower representation* (x^δ - LRep for short) and, respectively, a x^δ -*strictly lower representation* (x^δ - SLRep for short) if, for every $i = 1, \dots, r$, it holds $\varphi(x^{\eta_i + \alpha_i}) \leq \varphi(x^\delta)$ and respectively $\varphi(x^{\eta_i + \alpha_i}) < \varphi(x^\delta)$.

We observe that, as an obvious consequence of the definition of reduction procedure, if $g \xrightarrow{\mathcal{F}\mathcal{J}} h$, then $g - h$ has a representation by $\tau\mathcal{F}$ given by the steps of reduction (summing up the coefficients of each element of $\tau\mathcal{F}$ used more than once during the reduction).

Proposition 6.2. Let \mathcal{F} be a marked set over a weakly noetherian RS \mathcal{J} and let $g \in \mathcal{P}$.

- i) There exists a reduced form l of f obtained by \mathcal{F} and $g - l$ has a representation by $\tau\mathcal{F}$.
- ii) If \mathcal{J} has disjoint cones, then there is only one polynomial l (the canonical form of g) with $\text{Supp}(l) \subset N(J)$ and $g - l \in \langle \tau\mathcal{F} \rangle$; moreover, there is a unique representation of $g - l$ by $\tau\mathcal{F}$.
- iii) If \mathcal{J} is noetherian (with ordering function φ) and, for a reduced polynomial l , $g - l$ has a representation $\sum_{i=1}^r c_i x^{\gamma_i} f_{\alpha_i}$ by $\tau\mathcal{F}$ with all distinct heads $x^{\gamma_i + \alpha_i}$, then $g \xrightarrow{\mathcal{F}\mathcal{J}}_\star l$ and, for each i , $\varphi(x^{\gamma_i + \alpha_i}) \leq \varphi(x^\delta)$ for some $x^\delta \in \text{Supp}(g)$.
Vice versa, from $g \xrightarrow{\mathcal{F}\mathcal{J}} l$ one deduces that $g - l$ has a representation $\sum_{i=1}^r c_i x^{\gamma_i} f_{\alpha_i}$ by $\tau\mathcal{F}$ with all distinct heads s.t. for each i it holds $\varphi(x^{\gamma_i + \alpha_i}) \leq \varphi(x^\delta)$ for some $x^\delta \in \text{Supp}(g)$.
- iv) In the same hypotheses and setting of **iii**), if g is a term x^δ , then $x^\delta - l$ has a x^δ - LRep by $\tau\mathcal{F}$.

Proof. i) follows from the definition of $\xrightarrow{\mathcal{F}\mathcal{J}}$ and the weakly noetherianity of \mathcal{J} .

In order to prove ii) we observe that \mathcal{J} is in fact noetherian, since a RS with disjoint cones has no proper substructures. Then, let $\varphi: \mathcal{T} \rightarrow (W, <)$ be an ordering function.

Consider two reduced polynomials l, l' such that $g - l, g - l' \in \langle \tau\mathcal{F} \rangle$ and take some representations $g - l = \sum_{i=1}^r c_i x^{\gamma_i} f_{\alpha_i}$ and $g - l' = \sum_{i=1}^r d_i x^{\gamma_i} f_{\alpha_i}$ in $\tau\mathcal{F}$; we may suppose that the indices of the two summations are the same, possibly adding some zeroes.

We have then $l - l' = \sum_{i=1}^r (d_i - c_i) x^{\gamma_i} f_{\alpha_i}$ and we deduce that $c_i = d_i$ for $i = 1, \dots, r$. If, in fact, this were not true, we could choose a maximal element in the set $\{\varphi(x^{\gamma_i + \alpha_i}), i = 1, \dots, r, c_i - d_i \neq 0\}$: suppose it is $\varphi(x^{\gamma_1 + \alpha_1})$. Then $x^{\gamma_1 + \alpha_1}$ appears in the support of $\sum_{i=1}^r (d_i - c_i) x^{\gamma_i} f_{\alpha_i}$: indeed this term is different from $x^{\gamma_i + \alpha_i}$ for $i = 2, \dots, r$, since by hypothesis \mathcal{J} has disjoint cones, and it does not appear in the support of $x^{\gamma_i} f_{\alpha_i} - x^{\gamma_i + \alpha_i}$ for some $i = 1, \dots, r$, by maximality. We get then a contradiction since the support of $l - l'$ is contained in $N(J)$. Then $c_i = d_i$ and $l = l'$.

In order to prove iii), we proceed by induction on the number r of the summands. If $r = 1$ then $g = l + c_1 x^{\gamma_1} f_{\alpha_1}$, and $x^{\gamma_1 + \alpha_1}$ necessarily appears in $\text{Supp}(g)$, since it cannot coincide neither with a term in the support of l nor with a term of $x^{\gamma_1} f_{\alpha_1} - x^{\gamma_1 + \alpha_1}$. We can get l from g via a base reduction step on the term $x^{\gamma_1 + \alpha_1}$ using f_{α_1} .

Setting $\delta := \gamma_1 + \alpha_1$, we trivially have $\varphi(x^{\gamma_1 + \alpha_1}) \leq \varphi(x^\delta)$, $x^\delta \in \text{Supp}(g)$.

Moreover, each term x^β in the support of $c_1 x^{\gamma_1} f_{\alpha_1}$ satisfies $\varphi(x^\beta) \leq \varphi(x^\delta)$ since each

term $x^\gamma \in \text{Supp}(f_{\alpha_1} \setminus \{x^{\alpha_1}\})$ satisfies $\varphi(x^{\gamma_1+\gamma}) < \varphi(x^{\gamma_1+\alpha_1}) \leq \varphi(x^\delta)$.

Suppose by inductive hypothesis that the assertion is true in the case in which we have $r - 1$ summands.

We can suppose that $\varphi(x^{\gamma_r+\alpha_r})$ is maximal in the set $\{\varphi(x^{\gamma_i+\alpha_i}), i = 1, \dots, r\}$ and so it is also maximal in $\{\varphi(x^\epsilon) \mid x^\epsilon \in \text{Supp}(x^{\gamma_i} f_{\alpha_i}), i = 1, \dots, r\}$.

Then $x^{\gamma_r+\alpha_r}$ appears in the support of $\sum_{i=1}^r c_i x^{\gamma_i} f_{\alpha_i}$ and so also in the support of g (remember that $\text{Supp}(l) \subset \text{N}(J)$).

We execute the first reduction step on g choosing exactly that term and setting $g' = g - c_r x^{\gamma_r} f_{\alpha_r}$.

Setting $\delta := \gamma_r + \alpha_r$, we trivially have, for each i ,

$$\varphi(x^{\gamma_i+\alpha_i}) \leq \varphi(x^{\gamma_r+\alpha_r}) = \varphi(x^\delta), x^\delta \in \text{Supp}(g).$$

Thus we obtain $g' - l = \sum_{i=1}^{r-1} c_i x^{\gamma_i} f_{\alpha_i}$ and we conclude by inductive hypothesis.

The last statement immediately follows from the fact that \mathcal{J} is ordered and also from the property of the reduction procedure.

Last item is a consequence of the previous one and of Definition 6.1. \square

Corollary 6.3. *Let \mathcal{F} be a marked set over a weakly noetherian RS \mathcal{J} . Then*

$$\langle \tau\mathcal{F} \rangle + \langle \text{N}(J) \rangle = \mathcal{P}.$$

If, moreover, \mathcal{J} has disjoint cones, then

$$\langle \tau\mathcal{F} \rangle \oplus \langle \text{N}(J) \rangle = \mathcal{P}.$$

In particular, take $x^\eta \in \mathcal{T}$, $g, l \in \mathcal{P}$ s.t. $\text{Supp}(l) \subseteq \text{N}(J)$ and $\varphi(x^\gamma) \leq \varphi(x^\eta)$, for every $x^\gamma \in \text{Supp}(g)$. Then

$$g - l \in \langle \tau\mathcal{F} \rangle \iff g \xrightarrow{\mathcal{F}\mathcal{J}}_* l \iff g - l \text{ has a } \varphi(x^\eta) - \text{LRep by } \tau\mathcal{F}.$$

Proof: The first assertion comes from Proposition 6.2. Indeed, $\forall g \in \mathcal{P}$, from $g = \sum_{i=1}^r c_i x^{\gamma_i} f_{\alpha_i} + l$ with $x^{\gamma_i} f_{\alpha_i} \in \tau\mathcal{F}$ and $\text{Supp}(l) \subset \text{N}(J)$ we deduce $g \in \langle \tau\mathcal{F} \rangle + \langle \text{N}(J) \rangle$. So $\langle \tau\mathcal{F} \rangle + \langle \text{N}(J) \rangle \supseteq \mathcal{P}$. The other implication is obvious.

For the second assertion it is then sufficient to prove that $\langle \tau\mathcal{F} \rangle \cap \langle \text{N}(J) \rangle = 0$ and this comes from Proposition 6.2 ii).

Now we prove the last assertion. If $g - l \in \langle \tau\mathcal{F} \rangle$, by 6.2 ii), then $g - l$ has a unique representation $\sum_{i=1}^r c_i x^{\gamma_i} f_{\alpha_i}$ by $\tau\mathcal{F}$; as \mathcal{J} has disjoint cones, the heads $x^{\gamma_i} f_{\alpha_i}$ are distinct. By 6.2 iii) we obtain $g \xrightarrow{\mathcal{F}\mathcal{J}}_* l$ and $\varphi(x^{\gamma_i} f_{\alpha_i}) \leq \varphi(x^\delta)$ for some $x^\delta \in \text{Supp}(g)$; then $\varphi(x^{\gamma_i} f_{\alpha_i}) \leq \varphi(x^\eta)$, namely $\sum_{i=1}^r c_i x^{\gamma_i} f_{\alpha_i}$ is a $\varphi(x^\eta) - \text{LRep}$.

The other implications are obvious. \diamond

In what follows, we will use the second assertion of Proposition 6.2 (iii). Indeed, if one wants to use induction in proofs, it will be useful to consider the fact that not only a certain polynomial g is in $\langle \tau\mathcal{F} \rangle$, but also that g can be written as a linear combination of elements in $\tau\mathcal{F}$ whose heads satisfy the property underlined in (iii).

The following two examples show that the hypotheses of the various points of Proposition 6.2 are necessary. Point ii) does not necessarily hold if \mathcal{J} has non-disjoint cones. Moreover, the conditions $g - l \in \langle \tau\mathcal{F} \rangle$ and $\text{Supp}(l) \subset \text{N}(J)$ do not necessarily imply that $g \xrightarrow{\mathcal{F}\mathcal{J}} l$.

Example 6.4. In $A[x, y]$, let $\mathcal{J} = (M = \{x^2, xy\}, \{\lambda_{x^2} = \lambda_{xy} = \{x\}\}, \{\tau_{x^2} = \mathcal{T}, \tau_{xy} = \{y^k, xy^k, k \in \mathbb{N}\}\})$; notice that \mathcal{J} is noetherian, since it is coherent with any degree compatible term order, and it has not disjoint cones ($x^2y = x^2 \cdot y = xy \cdot x$). Let moreover $\mathcal{F} = \{f_{x^2} = x^2 - x, f_{xy} = xy\}$ and $g = x^2y - xy$. For each reduction $g \xrightarrow{\mathcal{F}\mathcal{J}}_* l$ we have $l = 0$, but g has two representations of the form of Proposition 6.2 ii): $g = yf_{x^2} = xf_{xy} - f_{xy}$.

Example 6.5. Consider the RS $\mathcal{J} = (M = \{x^2, xy, y^2\}, \{\lambda_{x^2} = \lambda_{y^2} = \lambda_{xy} = \{1\}\}, \{\tau_{x^2} = \tau_{y^2} = \tau_{xy} = \mathcal{T}\})$ and the marked set $\mathcal{F} = \{x^2 - 1, xy, y^2\}$ in $A[x, y]$.

For $g = y^3$ and $l = y$ we have $g - l = yf_{x^2} - xf_{xy} + yf_{y^2} \in \langle \tau\mathcal{F} \rangle$ and $\text{Supp}(l) \subset \mathbf{N}(J)$, but g has only one possible complete reduction $g \xrightarrow{f_{y^2}} 0$; therefore, $g \rightarrow_* l$ does not hold. Notice that $g = y^3 = yf_{y^2} \in \langle \tau\mathcal{F} \rangle$, whereas $l = -yf_{x^2} + xf_{xy} \in \langle \tau\mathcal{F} \rangle \cap \langle \mathbf{N}(J) \rangle$ is exactly the S-polynomial $S(f_{x^2}, f_{xy})$ (see Remark 8.3).

7. THE CONFLUENCY MATTER

Definition 7.1. Let \mathcal{F} be a marked set over a weakly noetherian RS \mathcal{J} . If for each polynomial g there is one and only one l s.t. $g \xrightarrow{\mathcal{F}\mathcal{J}}_* l$, then we call $\xrightarrow{\mathcal{F}\mathcal{J}}$ *confluent*.

We call \mathcal{J} *confluent* if for each marked set \mathcal{F} over \mathcal{J} , the reduction procedure $\xrightarrow{\mathcal{F}\mathcal{J}}$ is confluent.

This section is dedicated to the characterization of confluency. We will also identify a set of conditions which ensure confluency of a marked set.

The most significant case of confluent RS is the one presented in the following

Proposition 7.2. Let $\mathcal{J} = (M, \lambda, \tau)$ be a weakly noetherian RS. If \mathcal{J} has disjoint cones, then it is noetherian and confluent.

Proof. By Lemma 5.2 we know that \mathcal{J} is noetherian. We prove that each marked set \mathcal{F} over \mathcal{J} is confluent. For each $g \in \mathcal{P}$ there exists l s.t. $g \xrightarrow{\mathcal{F}\mathcal{J}}_* l$. If it holds also $g \xrightarrow{\mathcal{F}\mathcal{J}}_* l'$, then by Corollary 6.3, we would have $l' - l = (g - l) - (g - l') \in \langle \tau\mathcal{F} \rangle$, hence $l' - l = 0$. \square

We cannot invert the previous result in the general case: there are confluent reduction structures which do not have disjoint cones, as shown in the following example. Anyway, these are very specific cases and we cannot find relevant examples in literature.

Example 7.3. Every marked set over $\mathcal{J} = (M, \{\lambda_\alpha = \emptyset\}_{x^\alpha \in M}, \{\tau_\alpha = \mathcal{T}\}_{x^\alpha \in M})$ is monomial, then \mathcal{J} is clearly noetherian and confluent. Anyway, if M contains two terms at least, its cones are not disjoint!

Of course, a reduction structure consistent with a term ordering and maximal cones is both noetherian and with non-disjoint cones. In this “natural” setting confluency is related with ideal membership. On the other side, as we will see in the Section 13, in order to guarantee the confluence, the use of disjoint cones is widely spread. The counterpart, clearly, is that one has to transfer to a different procedure the resolution of *completion* and *membership test* problems. We will deal with this problem in next chapter.

Let \mathcal{J} be a weakly noetherian RS. Even if the cones in \mathcal{J} are not disjoint, we can “simulate” this property in the following way.

Let $\tilde{\tau} = \{\tilde{\tau}_\alpha, x^\alpha \in M\}$ be s.t. each $\tilde{\tau}_\alpha$ is a subset of τ_α ; in what follows we will denote by $\xrightarrow{\tilde{\tau}\mathcal{F}}$ the reduction process obtained by using only polynomials of $\tilde{\tau}\mathcal{F} := \{x^\eta f_\alpha \mid f_\alpha \in \mathcal{F}, x^\eta \in \tilde{\tau}_\alpha\}$.

Lemma 7.4. *Let $\mathcal{J} = (M, \lambda, \tau)$ be a weakly noetherian RS. Then, there is a list of sets of terms $\bar{\tau} = \{\bar{\tau}_\alpha\}_{x^\alpha \in M}$ with $\bar{\tau}_\alpha \subseteq \tau_\alpha$ s.t.*

- $\forall x^\alpha, x^{\alpha'} \in M, x^\alpha \neq x^{\alpha'}, \text{ one has } x^\alpha \bar{\tau}_\alpha \cap x^{\alpha'} \bar{\tau}_{\alpha'} = \emptyset$
- $\bigcup_{x^\alpha \in M} x^\alpha \bar{\tau}_\alpha = J$
- *for each marked set \mathcal{F} on \mathcal{J} , the reduction process $\xrightarrow{\bar{\tau}\mathcal{F}}$ is noetherian.*

Proof. By hypothesis there is a noetherian substructure $\mathcal{J}' = (M, \lambda, \tau')$ of \mathcal{J} , so $\tau'_\alpha \subseteq \tau_\alpha$.

Choose arbitrarily one of the subsets $\bar{\tau}_\alpha$ of τ'_α s.t. the first two conditions are satisfied. The last one is then immediate by the noetherianity of \mathcal{J}' . \square

We can then reinforce point iii) of Proposition 6.2.

Lemma 7.5. *Let $\mathcal{J} = (M, \lambda, \tau)$ be a weakly noetherian RS and let φ the ordering function of a noetherian substructure \mathcal{J}' . Then each polynomial g has a J -remainder l such that $g - l$ has a representation $\sum_j c_j x^{\gamma_j} f_{\alpha_j}$ by $\tau'\mathcal{F}$ with all distinct heads and $\varphi(x^{\gamma_j + \alpha_j}) < \varphi(x^\delta)$ for some $x^\delta \in \text{Supp}(g)$.*

Proof. It is sufficient to consider the reduction w.r.t. the polynomials of a set $\bar{\tau}\mathcal{F}$ as in Lemma 7.4. The consequent rewriting satisfies all the required conditions. Indeed, the polynomials in $\bar{\tau}\mathcal{F}$ have distinct heads by construction. Moreover, the reduction process $\xrightarrow{\bar{\tau}\mathcal{F}}$ can also be considered as obtained by $\xrightarrow{\mathcal{F}\mathcal{J}'}$ so also the second condition is satisfied by Proposition 6.2 iii). \square

Proposition 7.6. *Let \mathcal{F} be a marked set over a weakly noetherian RS \mathcal{J} and let \mathcal{J}' and $\bar{\tau}\mathcal{F}$ as in Lemma 7.4.*

Then $\bar{\tau}\mathcal{F}$ is a free set of generators of $\langle \bar{\tau}\mathcal{F} \rangle$ and $\forall g \in \mathcal{P}$, there exists a unique J -remainder l s.t. $g \xrightarrow{\bar{\tau}\mathcal{F}} l$; moreover $l = 0$ if and only if $g \in \langle \bar{\tau}\mathcal{F} \rangle$. Therefore

$$\langle \bar{\tau}\mathcal{F} \rangle \oplus \langle \mathbf{N}(J) \rangle = \mathcal{P}.$$

Proof. For every polynomial $g \in \mathcal{P}$, the J -remainder l exists since a reduction process w.r.t. $\xrightarrow{\bar{\tau}\mathcal{F}}$ can be applied to each term in J and this procedure terminate as $\bar{\tau}\mathcal{F}$ is a subset of $\tau'\mathcal{F}$.

Uniqueness is proved similarly to ii) of Proposition 6.2: notice that the elements of $\bar{\tau}\mathcal{F}$ have all distinct heads; moreover $\xrightarrow{\bar{\tau}\mathcal{F}}$ is noetherian since \mathcal{J}' is noetherian.

With the same argument one can also prove that the elements of $\bar{\tau}\mathcal{F}$ are linearly independent and the sum $\langle \bar{\tau}\mathcal{F} \rangle + \langle \mathbf{N}(J) \rangle$ is direct. \square

Theorem 7.7. *Let \mathcal{F} be a marked set over a weakly noetherian RS \mathcal{J} and let \mathcal{J}' and $\bar{\tau}$ as in Lemma 7.4. TFAE:*

- i) $\xrightarrow{\mathcal{F}\mathcal{J}}$ is confluent
- ii) $\langle \tau\mathcal{F} \rangle \oplus \langle \mathbf{N}(J) \rangle = \mathcal{P}$
- iii) $\langle \tau\mathcal{F} \rangle \cap \langle \mathbf{N}(J) \rangle = 0$
- iv) $\langle \tau\mathcal{F} \rangle = \langle \tau'\mathcal{F} \rangle = \langle \bar{\tau}\mathcal{F} \rangle$
- v) *for each $x^\eta f_\alpha \in \tau\mathcal{F} \setminus \bar{\tau}\mathcal{F}$ it holds $x^\eta f_\alpha \xrightarrow{\bar{\tau}\mathcal{F}}_\star 0$.*

- vi) for each $x^\eta f_\alpha \in \tau\mathcal{F}$, for each reduction $x^\eta f_\alpha \xrightarrow{\mathcal{F}\mathcal{J}'} l$ it holds $l = 0$;
- vii) for each $x^\eta f_\alpha \in \tau\mathcal{F}$, $x^{\eta'} f_{\alpha'} \in \tau'\mathcal{F}$ with $x^{\eta+\alpha} = x^{\eta'+\alpha'}$ it holds $x^\eta f_\alpha - x^{\eta'} f_{\alpha'} \xrightarrow{\mathcal{F}\mathcal{J}'} 0$.
- viii) for each $x^\eta f_\alpha \in \tau\mathcal{F}$, for each reduction $x^\eta f_\alpha \xrightarrow{\mathcal{F}\mathcal{J}} l$ it holds $l = 0$.
- ix) for each $x^\eta f_\alpha, x^{\eta'} f_{\alpha'} \in \tau\mathcal{F}$ with $x^{\eta+\alpha} = x^{\eta'+\alpha'}$, for each reduction $x^\eta f_\alpha - x^{\eta'} f_{\alpha'} \xrightarrow{\mathcal{F}\mathcal{J}} l$ it holds $l = 0$.

Proof. ii) \Leftrightarrow iii): the assertion trivially follows from Corollary 6.3.

iii) \Rightarrow i): notice that if $g \xrightarrow{\mathcal{F}\mathcal{J}} l$ and $g \xrightarrow{\mathcal{F}\mathcal{J}} l'$, then the difference $l - l'$ belongs to $\langle \tau\mathcal{F} \rangle \cap \langle \mathbf{N}(J) \rangle$, so $l - l' = 0$.

ii) \Leftrightarrow iv) \Leftrightarrow v) : follow from Lemma 7.6 and from the fact that by construction $\tau\mathcal{F} \supseteq \tau'\mathcal{F} \supseteq \bar{\tau}\mathcal{F}$.

viii) \Rightarrow vi) \Rightarrow v) are trivial, since the reductions by $\xrightarrow{\bar{\tau}\mathcal{F}}$ are particular cases of the ones by $\xrightarrow{\mathcal{F}\mathcal{J}'}$, that are particular cases of the ones by $\xrightarrow{\mathcal{F}\mathcal{J}}$. Notice that \mathcal{F} is weakly noetherian, so each polynomial has at least a total reduction.

i) \Rightarrow viii) is again obvious; indeed every polynomial $x^\eta f_\alpha \in \tau\mathcal{F}$ has at least the complete reduction $x^\eta f_\alpha \xrightarrow{\mathcal{F}\mathcal{J}} x^\eta f_\alpha - x^\eta f_\alpha = 0$

As a consequence of what proved so far, the conditions i), ii), iii), iv), v), vi), viii) are equivalent.

iii) \Rightarrow ix): it is sufficient to observe that in the hypotheses ix) the polynomial $x^\eta f_\alpha - x^{\eta'} f_{\alpha'}$ belongs to $\tau\mathcal{F}$ and l is in the intersection $\langle \tau\mathcal{F} \rangle \cap \langle \mathbf{N}(J) \rangle$.

ix) \Rightarrow vii) directly follows by the same argument used to prove “viii) \Rightarrow vi) \Rightarrow v)”.

We finally prove vii) \Rightarrow iv). By Proposition 6.2 i), condition vii) implies $\langle \tau\mathcal{F} \rangle \subseteq \langle \tau'\mathcal{F} \rangle$. Then, it is sufficient to prove that $\langle \tau'\mathcal{F} \rangle \subseteq \langle \bar{\tau}\mathcal{F} \rangle$, the opposite inclusions being obvious.

Assume by contradiction that the set $\tau'\mathcal{F} \setminus \langle \bar{\tau}\mathcal{F} \rangle$ is not empty and chose in it an element $x^\eta f_\alpha$ with minimal $\varphi(x^{\eta+\alpha})$, where φ is an ordering function for the noetherian RS \mathcal{J}' . Let, moreover, $x^{\eta'} f_{\alpha'}$ the only element in $\bar{\tau}\mathcal{F}$ such $x^{\eta+\alpha} = x^{\eta'+\alpha'}$: we may apply vii) to these two elements (as $x^\eta f_\alpha \in \tau'\mathcal{F} \subseteq \tau\mathcal{F}$ and $x^{\eta'} f_{\alpha'} \in \bar{\tau}\mathcal{F} \subseteq \tau'\mathcal{F}$) finding a reduction $x^\eta f_\alpha - x^{\eta'} f_{\alpha'} \xrightarrow{\mathcal{F}\mathcal{J}'} 0$.

We observe that every term $x^\gamma \in \text{Supp}(x^\eta f_\alpha - x^{\eta'} f_{\alpha'})$ is either in $\text{Supp}(x^\eta f_\alpha - x^{\eta+\alpha})$ or in $\text{Supp}(x^{\eta'} f_{\alpha'} - x^{\eta'+\alpha'})$. In both cases, $\varphi(x^\gamma) < \varphi(x^{\eta+\alpha}) = \varphi(x^{\eta'+\alpha'})$.

Then, by Corollary 6.3, the polynomial $x^\eta f_\alpha - x^{\eta'} f_{\alpha'}$ has a $\varphi(x^{\eta+\alpha})$ - SLRep in $\tau'\mathcal{F}$ of the type $\sum_{i=1}^r c_i x^{\gamma_i} f_{\alpha_i}$. By the minimality of $\varphi(x^{\eta+\alpha})$ we deduce that the summands $x^{\gamma_i} f_{\alpha_i}$ belong to $\langle \bar{\tau}\mathcal{F} \rangle$, hence also $x^\eta f_\alpha - x^{\eta'} f_{\alpha'}$ does. This is the wanted contradiction, as $x^\eta f_\alpha \notin \langle \bar{\tau}\mathcal{F} \rangle$ and $x^{\eta'} f_{\alpha'} \in \bar{\tau}\mathcal{F}$. \square

8. THE IDEAL MEMBERSHIP PROBLEM

In this section we will see which conditions have to be satisfied by a marked set \mathcal{F} over a RS \mathcal{J} in order that the rewriting rule $\xrightarrow{\mathcal{F}\mathcal{J}}$ give a criterion which is equivalent to the belonging to the ideal generated by \mathcal{F} , i.e. in order that

$$g \equiv g' \pmod{(\mathcal{F})} \iff \forall g \xrightarrow{\mathcal{F}\mathcal{J}} l \text{ and } \forall g' \xrightarrow{\mathcal{F}\mathcal{J}} l', \text{ it holds } l = l'$$

is satisfied.

We can observe that \mathcal{J} has necessarily to be noetherian (or, at least, weakly noetherian); indeed, if there is a polynomial g without complete reductions, the reduction cannot allow us to establish that g belongs to (\mathcal{F}) .

The ideal membership can be reformulated with the following definition, which constitutes a central point for the whole theory.

Definition 8.1. A marked set \mathcal{F} over a RS \mathcal{J} is called *marked basis* if $\mathbf{N}(J)$ is a free set of generators for $A[x_1, \dots, x_n]/(\mathcal{F})$ as A -module, i.e. if it holds

$$(\mathcal{F}) \oplus \langle \mathbf{N}(J) \rangle = \mathcal{P}.$$

Theorem 8.2. Let $\mathcal{J} = (M, \lambda, \tau)$ be a weakly noetherian RS and let $\mathcal{J}' = (M, \lambda, \tau')$ be a noetherian substructure. Let $\bar{\tau}$ be as in Lemma 7.4. Moreover, let \mathcal{F} be a marked set over \mathcal{J} .

If \mathcal{F} is a marked basis, then $\xrightarrow{\mathcal{F}\mathcal{J}}$ is confluent.

On the other hand, if we suppose that $\xrightarrow{\mathcal{F}\mathcal{J}}$ is confluent, then \mathcal{F} is a marked basis if and only if one of the following equivalent conditions holds:

- i) $(\mathcal{F}) = \langle \bar{\tau}\mathcal{F} \rangle$
- ii) $(\mathcal{F}) = \langle \tau'\mathcal{F} \rangle$
- iii) $(\mathcal{F}) = \langle \tau\mathcal{F} \rangle$

iv) for each $x^\alpha \in M$ and each $x^\gamma \notin \bar{\tau}_\alpha$ it holds $x^\gamma f_\alpha \xrightarrow{\bar{\tau}\mathcal{F}}_\star 0$

v) for each $x^\alpha \in M$ and $x^\gamma \notin \tau'_\alpha$ it holds $x^\gamma f_\alpha \xrightarrow{\mathcal{F}\mathcal{J}'}_\star 0$

vi) for each $x^\alpha \in M$ and $x^\gamma \notin \tau_\alpha$ it holds $x^\gamma f_\alpha \xrightarrow{\mathcal{F}\mathcal{J}}_\star 0$.

Proof. If \mathcal{F} is a marked basis, we have $\langle \tau\mathcal{F} \rangle \cap \langle \mathbf{N}(J) \rangle \subseteq (\mathcal{F}) \cap \langle \mathbf{N}(J) \rangle = 0$; then $\xrightarrow{\mathcal{F}\mathcal{J}}$ is confluent by Theorem 7.7 iii) \Rightarrow i).

Suppose now that $\xrightarrow{\mathcal{F}\mathcal{J}}$ is confluent. The equivalence between the condition i) and the fact that \mathcal{F} is a marked basis comes from Proposition 7.6.

In order to conclude, it is sufficient to prove that, under the given hypotheses that \mathcal{J} is weakly noetherian and $\xrightarrow{\mathcal{F}\mathcal{J}}$ is confluent, the enumerated conditions are equivalent.

The conditions i), ii), iii) are equivalent by Theorem 7.7.

Moreover, iv), v), vi) are equivalent since the confluence of $\xrightarrow{\mathcal{F}\mathcal{J}}$ grants also the confluence of $\xrightarrow{\mathcal{F}\mathcal{J}'}$ and $\xrightarrow{\bar{\tau}\mathcal{F}}$. Notice that in each of the three conditions iv), v), vi) the restriction on x^γ could be omitted; indeed, if for instance $x^\gamma \in \tau_\alpha$, then by a single step of reduction on $x^{\gamma+\alpha}$ we obtain $x^\gamma f_\alpha \xrightarrow{\mathcal{F}\mathcal{J}}_\star x^\gamma f_\alpha - x^\gamma f_\alpha = 0$.

We conclude proving that i) \Leftrightarrow iv). This is consequence of Proposition 7.6 and of the above remark about x^γ . \square

Remark 8.3. We can reformulate the characterizations of confluence of Theorem 7.7 and the ones of marked bases of Theorem 8.2 using the reduction w.r.t. polynomials of the form $x^\eta f_\alpha - x^{\eta'} f_{\alpha'}$ with $x^{\eta+\alpha} = x^{\eta'+\alpha'}$. Notice, anyway, that we do not have to examine only the S-polynomials $S(f_{\alpha'}, f_\alpha) := x^\eta f_\alpha - x^{\eta'} f_{\alpha'}$, i.e. the cases in which $x^{\eta+\alpha} = x^{\eta'+\alpha'} = \text{lcm}(x^\alpha, x^{\alpha'})$, but it is necessary to consider also all their multiples. Then, in order to apply these results we would execute the reduction of an infinite number of polynomials.

In Section 9 we will prove that for every weakly noetherian reduction structure there exists a finite set of controls using reductions that are sufficient to ensure that a marked set is a marked basis. However, this result is non-constructive.

For this reason, for practical purposes, it is necessary to consider reduction structures with particular properties, allowing to execute the verification of confluency or of being a marked basis with a given, finite (possibly small) number of reductions. We will examine two special cases in Sections 10 and 12.

In the case of homogeneous structures, due to Gotzmann Persistence [32] and Macaulay Estimate of Growth [33, Theorem 3.3], the controls one has to perform can be limited to the polynomials whose degree is bounded from above by $1+r$, where r is the maximum between the maximal degree of terms in M and the Castelnuovo-Mumford Regularity of the monomial ideal $J = (M)$.

A similar upper bound on the degree of polynomials involved in a sufficient set of controls appears also in the affine case in [5, Theorems 5.1 and 5.4]; indeed, those affine marked sets are marked bases if the following refinement of the condition *ii*) of Theorem 8.2 holds: $(\mathcal{F})_{\leq t} = \langle \tau \mathcal{F} \rangle_{\leq t}$ for some integers $t \leq r+1$.

Finally the recent result proved by [62] gives a further bound:

$$(\mathcal{F})_{\leq t} = \langle \tau \mathcal{F} \rangle_{\leq t} \text{ for all } t \geq 2 \left(\frac{d^2}{2} + d \right)^{2^{n-1}} + \sum_{j=0}^{n-1} (ud)^{2^j}$$

where $d = \max \deg(f : f \in \mathcal{F})$ and $u = \# \mathcal{F}$.

In the above quoted cases we should perform a finite (but in general not small) number of controls.

9. FUNCTORIALITY OF MARKED BASES

There are at least two functors from the category of rings to the category of sets that is natural to associate to a reduction structure $\mathcal{J} = (M, \lambda, \tau)$ in \mathcal{T} .

The functor of marked sets on \mathcal{J}

$$\underline{\mathbf{Ms}}_{\mathcal{J}} : \underline{\mathbf{Ring}} \rightarrow \underline{\mathbf{Set}}$$

that associates to any ring A the set $\underline{\mathbf{Ms}}_{\mathcal{J}}(A) := \{\text{marked sets } \mathcal{F} \text{ on } \mathcal{J}\}$ and to any morphism $\sigma : A \rightarrow B$ the map

$$\begin{aligned} \underline{\mathbf{Ms}}_{\mathcal{J}}(\sigma) : \underline{\mathbf{Ms}}_{\mathcal{J}}(A) &\longrightarrow \underline{\mathbf{Ms}}_{\mathcal{J}}(B) \\ \mathcal{F} &\longmapsto \sigma(\mathcal{F}). \end{aligned}$$

We observe that this functor is well defined since the coefficient of the distinguished term x^α in each marked polynomial $f_\alpha \in \mathcal{F}$ is the unit element of the coefficient ring.

Under the assumption that \mathcal{J} has finite tails, this functor is clearly represented by the affine space \mathbb{A}^N , where $N := \sum_{x^\alpha \in M} |\lambda_\alpha|$.

More interesting is the functor of the marked bases on \mathcal{J}

$$\underline{\mathbf{Mf}}_{\mathcal{J}} : \underline{\mathbf{Ring}} \rightarrow \underline{\mathbf{Set}}$$

that associates to any ring A the set $\underline{\mathbf{Mf}}_{\mathcal{J}}(A) := \{\text{marked bases } \mathcal{F} \text{ on } \mathcal{J}\}$ and to any morphism $\sigma : A \rightarrow B$ the map

$$\begin{aligned} \underline{\mathbf{Mf}}_{\mathcal{J}}(\sigma) : \underline{\mathbf{Mf}}_{\mathcal{J}}(A) &\longrightarrow \underline{\mathbf{Mf}}_{\mathcal{J}}(B) \\ \mathcal{F} &\longmapsto \sigma(\mathcal{F}). \end{aligned}$$

Notice that $\sigma(\mathcal{F})$ is indeed a marked basis in \mathcal{P}_B , as can be shown by applying $-\otimes_A B$ to $(\mathcal{F})_A \oplus \langle \mathbf{N}(J) \rangle_A = \mathcal{P}_A$.

Under the assumptions that \mathcal{J} has finite tails and is weakly noetherian, also this second functor turns out to be representable by an affine scheme, that can be obtained in the following way.

Let C be a set of variables $\{C_{\alpha\gamma}, \mid x^\alpha \in M \text{ and } x^\gamma \in \lambda_\alpha\}$ and consider in the ring $\mathbb{Z}[C]$ the marked set $\mathbf{F} := \{f_\alpha := x^\alpha + \sum_{x^\gamma \in \lambda_\alpha} C_{\alpha\gamma} x^\gamma \mid x^\alpha \in M\}$ over \mathcal{J} .

Then, compute all the complete reductions $x^\eta f_\alpha \xrightarrow{\mathbf{F}\mathcal{J}}_* l$ for every $x^\alpha \in M$ and $x^\eta \in \mathcal{T}$ and collect in a set $\mathcal{R} \subset \mathbb{Z}[C]$ the coefficients of the monomials $x^\eta \in \mathbf{N}((M))$ of all the reduced polynomials l . By Proposition 7.7 and Theorem 8.2 the marked set $\pi(\mathbf{F})$, where $\pi: \mathbb{Z}[C] \rightarrow \mathbb{Z}[C]/(\mathcal{R})$, is in fact a marked basis.

The functor $\mathbf{Mf}_{\mathcal{J}}$ is represented by the scheme $\mathbf{Mf}_{\mathcal{J}}(\mathbb{Z}) := \text{Spec}(\mathbb{Z}[C]/(\mathcal{R}))$ (for a detailed proof see [42]: the arguments presented there also apply in our, more general, framework).

Corollary 9.1. *Let $\mathcal{J} = (M, \lambda, \tau)$ be a weakly noetherian RS with finite tails.*

Then, there exists a finite subset $G \subset \mathcal{T} \times M$ such that for every marked set \mathcal{F} on \mathcal{J} TFAE:

- 1) \mathcal{F} is a marked basis
- 2) for all $(x^\eta, x^\alpha) \in G$ and for all reduction $x^\eta f_\alpha \xrightarrow{\mathbf{F}\mathcal{J}}_* l$ it holds $l = 0$.

Proof. By the noetherianity of the ring $\mathbb{Z}[C]$ there exists a finite set $\mathcal{R}' \subset \mathcal{R}$ that generates the ideal (\mathcal{R}) . For every element $r \in \mathcal{R}'$ let us choose $x^\eta \in \mathcal{T}$ and $f_\alpha \in \mathbf{F}$ and a reduction $x^\eta f_\alpha \xrightarrow{\mathbf{F}\mathcal{J}}_* l$ s.t. r is a coefficient in l ; then let us collect in G the pairs (x^η, x^α) .

The thesis is a direct consequence of the fact that $\mathcal{F} := \{f_\alpha + \sum_{x^\gamma \in \lambda_\alpha} c_{\alpha\gamma} x^\gamma, x^\alpha \in M\} \subset \mathcal{P}_A$ is a marked basis on \mathcal{J} if and only if the morphism $\sigma: \mathbb{Z}[C] \rightarrow A$ given by $\sigma(C_{\alpha\gamma}) = c_{\alpha\gamma}$ factorizes through $\mathbb{Z}[C]/(\mathcal{R}) = \mathbb{Z}[C]/(\mathcal{R}')$. \square

10. MAXIMAL AND DISJOINT CONES

In the usual reduction procedure w.r.t. a set of marked polynomials, one admits to rewrite any multiple of x^α with the marked polynomial f_α whose head is x^α . In our language, every term in \mathcal{T} is considered as multiplicative for each $x^\alpha \in M$: these are the structures we call *with maximal cones*.

If such a RS $\mathcal{J} = (M, \lambda, \tau)$ is noetherian we already remarked that it must be necessarily consistent with a term order by Theorem 5.9. Then the marked bases over \mathcal{J} are Gröbner bases. Moreover, for a set \mathcal{F} marked over \mathcal{J} the fact of being a marked basis and the confluency of $\xrightarrow{\mathbf{F}\mathcal{J}}$ are equivalent, since (\mathcal{F}) and $\langle \tau\mathcal{F} \rangle$ coincide by construction.

It is a well known fact that in the Gröbner case, in order to check whether a marked set is a marked basis (*id est* a Gröbner basis) it is sufficient to perform a finite number of controls which can be deduced by the given data, namely Buchberger test/completion result states that a basis (in our language: a marked set) \mathcal{F} is Gröbner (in our language: a marked basis) if and only if each element in the set of all S-polynomials

$$\left\{ S(f_{\alpha'}, f_\alpha) := \frac{\text{lcm}(x^\alpha, x^{\alpha'})}{x^\alpha} f_\alpha - \frac{\text{lcm}(x^\alpha, x^{\alpha'})}{x^{\alpha'}} f_{\alpha'} : x^\alpha, x^{\alpha'} \in M \right\}$$

between two elements of \mathcal{F} , reduces to 0.

Thus we do not need to check their multiples as in the general case.

Being a well known theory, we will not treat it in the usual way, but we change our point of view.

Notice that if \mathcal{J} is a RS and \mathcal{J}' is one of its substructures, a set \mathcal{F} - marked over \mathcal{J} - is also marked over \mathcal{J}' ; moreover, the concept of marked basis depends only on \mathcal{F} and it does not depend on the RS over which we consider it as a marked set. Thus we can construct a substructure \mathcal{J}' of \mathcal{J} having disjoint cones in order to characterize the marked bases over \mathcal{J} using the good properties of the RS with disjoint cones \mathcal{J}' . We propose here one of the possible ways to do so.

Lemma 10.1. *If $\mathcal{J} = (M, \lambda, \tau)$ is a RS with maximal cones, then there is a substructure $\mathcal{J}' = (M, \lambda, \tau')$ with disjoint cones.*

Proof. Consider the set $M = \{x^{\alpha_1}, \dots, x^{\alpha_s}\}$ and suppose that its terms are ordered in such a way that none of the x^{α_i} is multiple of a term with an index $< i$.

First of all, set $\tau'_{\alpha_1} := \tau$ then $\tau'_{\alpha_2} x^{\alpha_2} := x^{\alpha_2} \tau \setminus x^{\alpha_1} \tau$. Notice that τ'_{α_2} is an order ideal (in particular $1 \in \tau'_{\alpha_2}$) since $x^{\alpha_1} \nmid x^{\alpha_2}$.

By induction, after determining the multiplicative sets of the first r terms of M set $x^{\alpha_{r+1}} \tau'_{\alpha_{r+1}} := x^{\alpha_{r+1}} \tau \setminus \bigcup_{i=1}^r x^{\alpha_i} \tau'_{\alpha_i}$. \square

If \mathcal{J} is coherent with a term order \prec (hence noetherian) and \mathcal{J}' is a substructure with disjoint cones, then \mathcal{J}' is noetherian too. Moreover, due to Theorem 7.7, we can say that for each set \mathcal{F} marked over \mathcal{J}' the reduction process $\xrightarrow{\mathcal{F}\mathcal{J}'} l$ is noetherian and confluent. Indeed, the condition *ix*) is empty.

More in general, in the last part of this section, we will study the properties of noetherian reduction structures with disjoint cones, for which the following condition on the ordering function $\varphi: \mathcal{T} \rightarrow (E, >)$ holds:

$$(2) \quad \forall x^\delta, x^{\delta'}, x^\varepsilon \in \mathcal{T} : \varphi(x^\delta) > \varphi(x^{\delta'}) \Rightarrow \varphi(x^{\delta+\varepsilon}) > \varphi(x^{\delta'+\varepsilon}) \geq \varphi(x^\varepsilon).$$

This condition clearly holds if \mathcal{J}' is consistent with a term order \prec and we choose as ordering function $Id_{\mathcal{T}}: \mathcal{T} \rightarrow (\mathcal{T}, \prec)$.

Proposition 10.2. *Let \mathcal{J}' be a noetherian RS with disjoint cones and suppose also that its ordering function φ satisfies (2). If \mathcal{F} is a marked set over \mathcal{J}' and x^β is a term, TFAE:*

- i) *for each $x^\alpha \in M$, $x^\eta \notin \tau_\alpha$ s.t. $\varphi(x^{\eta+\alpha}) < \varphi(x^\beta)$, it holds $x^\eta f_\alpha \in \langle \tau' \mathcal{F} \rangle$*
- ii) *for each $x^\alpha \in M$, $x^\eta \notin \tau_\alpha$ s.t. $\varphi(x^{\eta+\alpha}) < \varphi(x^\beta)$, $x^\eta f_\alpha$ has a $\varphi(x^\beta)$ - SLRep.*
- iii) *for each S-polynomial*

$$S(f_{\alpha'}, f_\alpha) := \frac{lcm(x^\alpha, x^{\alpha'})}{x^\alpha} f_\alpha - \frac{lcm(x^\alpha, x^{\alpha'})}{x^{\alpha'}} f_{\alpha'}$$

s.t. $\frac{lcm(x^\alpha, x^{\alpha'})}{x^\alpha} \notin \tau_{\alpha'}$, $\frac{lcm(x^\alpha, x^{\alpha'})}{x^{\alpha'}} \in \tau_{\alpha'}$ and $\varphi(lcm(x^\alpha, x^{\alpha'})) < \varphi(x^\beta)$, $S(f_{\alpha'}, f_\alpha)$ has a $\varphi(x^\beta)$ - SLRep.

- iv) *for each S-polynomial*

$$S(f_{\alpha'}, f_\alpha) := \frac{lcm(x^\alpha, x^{\alpha'})}{x^\alpha} f_\alpha - \frac{lcm(x^\alpha, x^{\alpha'})}{x^{\alpha'}} f_{\alpha'}$$

s.t. $\frac{lcm(x^\alpha, x^{\alpha'})}{x^\alpha} \notin \tau_{\alpha'}$, $\frac{lcm(x^\alpha, x^{\alpha'})}{x^{\alpha'}} \in \tau_{\alpha'}$ and $\varphi(lcm(x^\alpha, x^{\alpha'})) < \varphi(x^\beta)$, it holds

$$S(f_{\alpha'}, f_\alpha) \xrightarrow{\mathcal{F}\mathcal{J}'} 0.$$

Proof. First of all we observe that in our hypotheses if $\varphi(x^{\eta+\alpha}) < \varphi(x^\beta)$ then also $\varphi(x^\delta) < \varphi(x^\beta)$ for every term $x^\delta \in \text{Supp}(x^\eta f_\alpha)$.

i) \Leftrightarrow ii) follows by Corollary 6.3.

ii) \Rightarrow iii) Consider an element $x^\eta f_\alpha$ satisfying the conditions of *ii)*. Since *i)* holds, it has a $\varphi(x^\beta)$ -SLRep; summing $-x^{\eta'} f_{\alpha'}$ we get a $\varphi(x^\beta)$ - SLRep of $x^\eta f_\alpha - x^{\eta'} f_{\alpha'}$.

iii) \Leftrightarrow iv) comes trivially from Corollary 6.3.

iii) + iv) \Rightarrow ii) Suppose by contradiction that the assertion is false and that x^β is a term with $\varphi(x^\beta)$ minimal among the ones not satisfying the condition. Consider $x^\eta f_\alpha$ with $\varphi(x^{\eta+\alpha}) < \varphi(x^\beta)$. Notice that, by hypothesis, the assertion is true in particular for $x^{\beta'} := x^{\eta+\alpha}$.

The assertion in *ii)* would immediately follow by *iii)* if for the only $x^{\eta'} f_{\alpha'}$ s.t. $x^{\eta+\alpha} = x^{\eta'+\alpha'}$ one has $\text{lcm}(x^\alpha, x^{\alpha'}) = x^{\eta+\alpha}$. So we must have $\text{lcm}(x^\alpha, x^{\alpha'}) = x^{\varepsilon+\alpha} = x^{\varepsilon'+\alpha'}$ with x^ε proper divisor of x^η . We can then apply *iv)* getting $x^\varepsilon f_\alpha - x^{\varepsilon'} f_{\alpha'} = S(f_{\alpha'}, f_\alpha) \xrightarrow{\mathcal{F}\mathcal{J}'} 0$. Notice that by (2) for each term x^γ in the support of $x^\varepsilon f_\alpha$ and of $x^{\varepsilon'} f_{\alpha'}$ it holds $\varphi(x^\gamma) < \varphi(x^{\varepsilon+\alpha})$. By Corollary 6.3, we have then $x^\varepsilon f_\alpha - x^{\varepsilon'} f_{\alpha'}$ has a $\varphi(x^{\varepsilon+\alpha})$ - SLRep, i.e. $x^\varepsilon f_\alpha - x^{\varepsilon'} f_{\alpha'} = \sum c_i x^{\gamma_i} f_{\alpha_i}$ with $\varphi(x^{\gamma_i+\alpha_i}) < \varphi(x^{\varepsilon+\alpha})$. Multiply this representation by $x^{\eta-\varepsilon}$. For each summand $x^{\eta-\varepsilon+\gamma_i} f_{\alpha_i}$ it holds $\varphi(x^{\eta-\varepsilon+\gamma_i+\alpha_i}) < \varphi(x^{\eta-\varepsilon+\varepsilon+\alpha}) = \varphi(x^{\eta+\alpha})$. By the assumption on the truth of our assertion with $x^{\beta'} = x^{\eta+\alpha}$, each polynomial $x^{\eta-\varepsilon+\gamma_i} f_{\alpha_i}$ has a $\varphi(x^{\eta+\alpha})$ - SLRep. We then get a $\varphi(x^{\eta+\alpha})$ - SLRep of $x^\eta f_\alpha - x^{\eta'} f_{\alpha'}$ so, summing to the two members $x^{\eta'} f_{\alpha'}$ we get a $\varphi(x^{\eta+\alpha})$ - LRep of $x^\eta f_\alpha$ since $x^{\eta'+\alpha'} \in \tau_{\alpha'}$. We conclude noticing that by hypothesis $\varphi(x^{\eta+\alpha}) < \varphi(x^\beta)$. \square

By the previous results and by Theorems 8.2 and 7.7 follows

Corollary 10.3. *Let \mathcal{J}' be a noetherian RS with disjoint cones and suppose also that (2) holds for the ordering function φ . Then for a marked set \mathcal{F} over \mathcal{J}' TFAE:*

- i) \mathcal{F} is a marked basis
- ii) $\forall x^\alpha, x^{\alpha'} \in M$ s.t. $\text{lcm}(x^\alpha, x^{\alpha'}) \in \text{cone}(x^{\alpha'})$ it holds $S(f_{\alpha'}, f_\alpha) \xrightarrow{\mathcal{F}\mathcal{J}'} 0$
- iii) $\forall x^\alpha, x^{\alpha'} \in M$ s.t. $x^{\gamma+\alpha} = \text{lcm}(x^\alpha, x^{\alpha'}) \in \text{cone}(x^{\alpha'})$ it holds $x^\gamma f_\alpha \xrightarrow{\mathcal{F}\mathcal{J}'} 0$.

For such reduction structures we can improve the characterization of marked bases given in Corollary 10.3 similarly to what done for Gröbner bases. We can verify that also in this context some of the known simplifications hold.

The “strategy” presented here exploits a substructure of \mathcal{J} with disjoint cones. Such a structure is inspired by (and generalizes) Gebauer-Moeller’s *Staggered linear bases*.

11. CRITERIA

Throughout this section, for notation simplicity, we will assume that the finite set M is enumerated as $\{x^{\alpha_1}, \dots, x^{\alpha_s}\}$ and we will relabel each element f_{α_i} in the related marked set

$$\mathcal{F} = \{f_\alpha\}_{x^\alpha \in M} = \{f_{\alpha_i}, 1 \leq i \leq s\}$$

as $f_i := f_{\alpha_i}, 1 \leq i \leq s$.

We will further assume to have performed the construction outlines in Lemma 10.1; in particular we have

$$\tau'_{\alpha_1} = \mathcal{T} \text{ and } x^{\alpha_{r+1}} \tau'_{\alpha_{r+1}} := x^{\alpha_{r+1}} \mathcal{T} \setminus \bigcup_{i=1}^r x^{\alpha_i} \tau'_{\alpha_i} \text{ for all } i;$$

Further we will assume that the elements of M are ordered so that

$$(3) \quad x^{\alpha_i} \mid x^{\alpha_j} \implies i < j.$$

We moreover denote

- for each $i, 1 \leq i \leq s, \mathbf{T}(i) := x^{\alpha_i}$,
- for each $i, j, 1 \leq i \leq s,$

$$\mathbf{T}(i, j) := \text{lcm}(\mathbf{T}(f_i), \mathbf{T}(f_j)) = \text{lcm}(x^{\alpha_i}, x^{\alpha_j})$$

and

- $S(i, j) := S(f_i, f_j) = \frac{\mathbf{T}(i, j)}{\mathbf{T}(j)} f_j - \frac{\mathbf{T}(i, j)}{\mathbf{T}(i)} f_i;$
- for each $i, j, k : 1 \leq i, j, k \leq s,$

$$\mathbf{T}(i, j, k) := \text{lcm}(\mathbf{T}(f_i), \mathbf{T}(f_j), \mathbf{T}(f_k)) = \text{lcm}(x^{\alpha_i}, x^{\alpha_j}, x^{\alpha_k}).$$

Lemma 11.1 (Möller). [50] *For each $i, j, k : 1 \leq i, j, k \leq s$ it holds*

$$\frac{\mathbf{T}(i, j, k)}{\mathbf{T}(i, k)} S(i, k) - \frac{\mathbf{T}(i, j, k)}{\mathbf{T}(i, j)} S(i, j) + \frac{\mathbf{T}(i, j, k)}{\mathbf{T}(k, j)} S(k, j) = 0.$$

Buchberger test/completion result states that a basis (in our language: a marked set) \mathcal{F} is Gröbner (in our language: a marked basis) if and only if each S-polynomial $S(i, j), i, j, 1 \leq i \leq s$, between two elements of \mathcal{F} , reduces to 0 and gave two criteria [11] to detect S-pairs which are “useless” in the sense that theoretical results prove that they reduce to 0, thus making useless the normal form computation. The First Criterion (Proposition 11.3) is based on a direct reformulation of trivial sygies, the Second applies Lemma 11.1.

We remark that the test/completion result given by Proposition 10.2.iv) allow to remove many useless S-pairs.

In fact, an S-polynomial $S(i, j)$ is not to be tested, and thus considered “useless”, if $\mathbf{T}(i, j) \notin \text{cone}(\mathbf{T}(i)) \cup \text{cone}(\mathbf{T}(j))$.

Example 11.2. Let us consider $M := \{x^{\alpha_i} : 1 \leq i \leq 3\}$ with

- $x^{\alpha_1} = \mathbf{T}(1) = xy, \tau_{\alpha_1} = \mathcal{T},$
- $x^{\alpha_2} = \mathbf{T}(2) = y^2, \tau_{\alpha_2} = \{y^i : i \in \mathbb{N}\},$
- $x^{\alpha_3} = \mathbf{T}(3) = x^2, \tau_{\alpha_3} = \{x^i : i \in \mathbb{N}\}$

and remark that

$$S(2, 3) = yS(1, 3) - xS(1, 2).$$

Note that

$$\mathbf{T}(2, 3) = x^2 y^2 \notin \text{cone}(\mathbf{T}(2)) \cup \text{cone}(\mathbf{T}(3)) = \{y^{i+2} : i \in \mathbb{N}\} \cup \{x^{i+2} : i \in \mathbb{N}\}$$

while

$$\frac{\mathbf{T}(1, 2)}{\mathbf{T}(1)} = y \in \tau_{\alpha_1} = \mathcal{T} \ni y \frac{\mathbf{T}(1, 3)}{\mathbf{T}(1)}$$

so we detect the “useless” pair $S(2, 3)$.

Naturally, we can prove in our setting Buchberger Second Criterion; we also can prove Buchberger First Criterion

Proposition 11.3. [11] (*Buchberger First Criterion*) *Under the hypotheses of Corollary 10.3 for \mathcal{F} being a marked basis it is not necessary to check whether the*

S-polynomials $S(f_{\alpha'}, f_{\alpha})$ s.t. $mcm(x^{\alpha}, x^{\alpha'}) = x^{\alpha+\alpha'}$ reduce to 0.

Proof. Suppose $mcm(x^{\alpha}, x^{\alpha'}) = x^{\alpha+\alpha'}$. Apply Proposition 10.2 choosing $x^{\beta} = x^{\alpha+\alpha'}$. If some of the requested controls is negative, \mathcal{F} is not a marked basis and we can conclude it without using $S(f_{\alpha'}, f_{\alpha})$. Otherwise, all the polynomials $x^{\epsilon} f_{\alpha'}$ with $\varphi(x^{\epsilon+\alpha'}) < \varphi(x^{\alpha'+\alpha})$ belong to $\langle \tau \mathcal{F} \rangle$.

Denoted $f_{\alpha} = x^{\alpha} - g_{\alpha}$ and $f_{\alpha'} = x^{\alpha'} - g_{\alpha'}$, it holds $x^{\alpha'} f_{\alpha} - x^{\alpha} f_{\alpha'} = g_{\alpha'} f_{\alpha} - g_{\alpha} f_{\alpha'}$. By definition of ordered RS, all the terms x^{δ} in the support of g_{α} are s.t. $\varphi(x^{\delta}) < \varphi(x^{\alpha})$, so by (2) we have $\varphi(x^{\delta+\alpha'}) < \varphi(x^{\alpha+\alpha'})$. Then $g_{\alpha} f_{\alpha'} \in \langle \tau \mathcal{F} \rangle$. Similarly we get $g_{\alpha'} f_{\alpha} \in \langle \tau \mathcal{F} \rangle$ and we conclude that their difference $S(f_{\alpha'}, f_{\alpha})$ is in $\langle \tau \mathcal{F} \rangle$. \square

Differently from Gröbner bases, it is not always true that the S-polynomial of two polynomials with coprime heads reduces to 0.

Example 11.4. Consider the RS with $M = \{x, y, xz\} \subset \mathcal{P} = A[x, y, z]$, $\tau_x = \mathcal{T}[x, y]$, $\tau_y = \tau_{xz} = \mathcal{T}$. Take $\mathcal{F} = \{f_x = x, f_y = y - z, f_{xz} = xz - z^2\}$. We will have then $y f_x, x f_y \in \tau \mathcal{F}$, but the only reduction of the S-polynomial $S(f_y, f_x) = y f_x - x f_y = xz \xrightarrow{f_{xz}}_{\star} z^2$ does not produce 0. The point, of course, is that (2) is not satisfied

Proposition 11.5. [11] (*Buchberger Second Criterion*) *Under the hypotheses of Corollary 10.3, for \mathcal{F} being a marked basis it is not necessary to control that $S(f_{\alpha'}, f_{\alpha''})$ reduces to 0 if we already checked $S(f_{\alpha'}, f_{\alpha})$ and $S(f_{\alpha''}, f_{\alpha})$, and $x^{\alpha} \mid mcm(x^{\alpha'}, x^{\alpha''})$.*

Proof. By hypothesis and Lemma 11.1 $S(f_{\alpha'}, f_{\alpha''}) = x^{\epsilon'} S(f_{\alpha'}, f_{\alpha}) - x^{\epsilon''} S(f_{\alpha''}, f_{\alpha})$ for some $x^{\epsilon'}, x^{\epsilon''} \in \mathcal{T}$. Apply Proposition 10.2 choosing $x^{\beta} = mcm(x^{\alpha'}, x^{\alpha''})$. If some of the requested controls is negative, \mathcal{F} is not a marked basis and we can conclude it without using $S(f_{\alpha'}, f_{\alpha''})$. Otherwise, we know that all the polynomials $x^{\epsilon} f_{\gamma}$ with $\varphi(x^{\epsilon+\gamma}) < \varphi(x^{\alpha'+\alpha})$ are in $\langle \tau \mathcal{F} \rangle$. By hypothesis we also know that $S(f_{\alpha'}, f_{\alpha}) \in \langle \tau \mathcal{F} \rangle$; so we can write it by a $\varphi(mcm(x^{\alpha+\alpha'}))$ -SLRep since for each term x^{δ} in the support of $S(f_{\alpha'}, f_{\alpha})$ one has $\varphi(x^{\delta}) < \varphi(mcm(x^{\alpha}, x^{\alpha'}))$. Then, multiplying the summands $x^{\eta_i} f_{\alpha_i}$ of this representation by $x^{\epsilon'}$, we get polynomials $x^{\epsilon'+\eta_i} f_{\alpha_i}$ belonging to $\langle \tau' \mathcal{F} \rangle$ since $\varphi(x^{\epsilon'+\eta_i+\alpha_i}) < \varphi(x^{\epsilon'} mcm(x^{\alpha}, x^{\alpha'})) = \varphi(mcm(x^{\alpha'}, x^{\alpha''}))$.

Then $x^{\epsilon'} S(f_{\alpha'}, f_{\alpha})$ is in $\langle \tau' \mathcal{F} \rangle$. Similarly we can obtain that $x^{\epsilon''} S(f_{\alpha''}, f_{\alpha})$ is in $\langle \tau' \mathcal{F} \rangle$ and we conclude. \square

Let us now enumerate the set of all S-pairs by a well-ordering \prec which preserves divisibility:

$$(4) \quad \mathbf{T}(i_1, j_1) \mid \mathbf{T}(i_2, j_2) \neq \mathbf{T}(i_1, j_1) \implies (i_1, j_1) \prec (i_2, j_2)$$

Corollary 11.6 (Buchberger). [11][51, II.Lemma 25.1.3] *Let*

$$\mathfrak{B} \subset \{\{i, j\}, 1 \leq i < j \leq s\}$$

be such that for each $\{i, j\}, 1 \leq i < j \leq s$, either

- $\mathbf{T}(i, j) = \mathbf{T}(i)\mathbf{T}(j)$ or
- there is $k, 1 \leq k \leq s$ such that
 - $\mathbf{T}(k) \mid \mathbf{T}(i, j)$ and
 - $\{i, k\} \prec \{i, j\}$
 - $\{k, j\} \prec \{i, j\}$.

Then under the hypotheses of Corollary 10.3 for \mathcal{F} being a marked basis it is sufficient to check whether the S -polynomials belonging to $\{\{i, j\}, 1 \leq i < j \leq s\} \setminus \mathfrak{B}$ for \mathcal{F} reduce to 0.

Proof. The proof is performed by induction according \prec : for each $i, j, 1 \leq i < j \leq s$, either

- $\{i, j\} \notin \mathfrak{B}$, and $S(i, j)$ reduces to 0 by assumption, or
- $\mathbf{T}(i)\mathbf{T}(j) = \mathbf{T}(i, j)$ and $S(i, j)$ reduces to 0 by Buchberger's First Criterion, or
- $S(i, j)$ reduces to 0 by Buchberger's Second Criterion, since by inductive assumption both $S(i, k)$ and $S(k, j)$ reduce to 0.

□

The following example shows that Corollary 10.3 can effectively apply the power granted by Möller Lemma and Buchberger's Corollary 11.6 only if the construction outlined in Lemma 10.1 is performed on the elements of M after having preliminarily ordered them so that (3) holds.

Example 11.7. Let $\mathcal{J} = (M = \{xy, xz, yz^2\}, \lambda, \tau)$ be the RS in $\mathcal{T} = \mathcal{T}[x, y, z]$ with disjoint cones given by $\tau_{xy} = \mathcal{T}[x, y]$, $\tau_{xz} = \mathcal{T}[x, z] \cup \mathcal{T}[x, y]$, $\tau_{yz^2} = \mathcal{T}[x, y, z]$ (and tails defined in any way such that \mathcal{J} be noetherian). In order to decide whether a marked set $\mathcal{F} = \{f_{xy}, f_{xz}, f_{yz^2}\}$ on \mathcal{J} is a basis according with Corollary 10.3 we should check the reductions of the three S -polynomials $S(f_{xz}, f_{xy}) = zf_{xy} - yf_{xz}$, $S(f_{yz^2}, f_{xy}) = z^2f_{xy} - xf_{yz^2}$, $S(f_{yz^2}, f_{xz}) = yzf_{xz} - xf_{yz^2}$. However, by Proposition 11.5 it is sufficient to check the first and either the second or the third pair, as both xy and xz divide $\text{lcm}(xy, yz^2) = \text{lcm}(xz, yz^2) = xyz^2$.

Note that we have

$$\begin{aligned} S(f_{yz^2}, f_{xy}) - zS(f_{xz}, f_{xy}) + S(f_{yz^2}, f_{xz}) &= \\ (z^2f_{xy} - xf_{yz^2}) - z(zf_{xy} - yf_{xz}) + (yzf_{xz} - xf_{yz^2}) &= 0 \end{aligned}$$

where $xf_{yz^2}, yzf_{xz} \notin \langle \tau' \mathcal{F} \rangle$ while $xf_{yz^2}, yf_{xz} \in \langle \tau' \mathcal{F} \rangle$; as a consequence we have

$$g := yf_{xz} \in \langle \tau' \mathcal{F} \rangle \not\Rightarrow zg = yzf_{xz} \in \langle \tau' \mathcal{F} \rangle$$

We further remark that the ordering of the elements of M which follows the construction proposed by Janet [38] has the negative aspect that the first element yz^2 to which, according the Staggered Basis construction outlined in Lemma 10.1, we associate $\tau_{yz^2} = \mathcal{T}$ is of higher degree then the other two elements.

This is the reason why we fail here to obtain the full effect of Möller Lemma.

It is well-known that the need of storing and ordering all pairs $\{i, j\}, 1 \leq i < j \leq s$, in order to extract \mathfrak{B} produces a bottleneck and is the weakness of Buchberger's Corollary 11.6; all efficient implementation of Buchberger Criteria have the ability of storing only "useful" pairs; our approach based on Corollary 10.3 shares then same property.

12. STABLY ORDERED REDUCTION STRUCTURES

Another case in which the control proving whether a marked set is a marked basis can be performed via a finite number of predetermined reductions is the case of *stably ordered reduction structures* that now we introduce.

In the following Section 13, we will examine some significant examples that are included in this case, such as border bases and Pommaret bases; we will see that for each of them we can consider term ordering free versions.

Definition 12.1. Let $\mathcal{J} = (M, \lambda, \tau)$ be a RS. We will say that \mathcal{J} is *stably ordered* if there is a function $\psi: \mathcal{T} \rightarrow (W, >)$ s.t. taken $x^\alpha, x^{\alpha'} \in M$:

StOr1: $\psi(x^\eta) > \psi(1)$ for each term $x^\eta \neq 1$

StOr2: $\psi(x^{\eta'}) > \psi(x^\eta)$ iff $\psi(x^{\eta'+\epsilon}) > \psi(x^{\eta+\epsilon})$

StOr3: if $x^{\eta+\alpha} \in \text{cone}(x^{\alpha'})$ and $x^\alpha \neq x^{\alpha'}$, then $\psi(x^\eta) > \psi(x^{\eta+\alpha-\alpha'})$

StOr4: if $x^\gamma \in \lambda_\alpha$, $x^\eta \in \tau_\alpha$ and $x^{\eta+\gamma} \in \text{cone}(x^{\alpha'})$ then $\psi(x^\eta) > \psi(x^{\eta+\gamma-\alpha'})$.

Remark that if we specialize ourselves to the case $W = \mathcal{T}$ and $\psi = \text{Id}_{\mathcal{T}}$, StOr1 and StOr2 grant that $<$ is a term ordering; via StOr2, StOr3 can be reformulated as

$$x^{\eta+\alpha} \in \text{cone}(x^{\alpha'}), x^\alpha \neq x^{\alpha'} \implies \psi(x^{\eta+\alpha'}) > \psi(x^{\eta+\alpha})$$

and StOr4 as

$$x^\gamma \in \lambda_\alpha, x^\eta \in \tau_\alpha, x^{\eta+\gamma} \in \text{cone}(x^{\alpha'}) \implies \psi(x^{\eta+\alpha'}) > \psi(x^{\eta+\gamma}).$$

Remark 12.2. A stably ordered RS \mathcal{J} has disjoint cones. Indeed, if there is a term $x^\delta \in \text{cone}(x^\alpha) \cap \text{cone}(x^{\alpha'})$ with $x^\alpha \neq x^{\alpha'}$, by StOr3 we would get the contradiction $\psi(x^{\delta-\alpha}) > \psi(x^{\delta-\alpha'})$ and also $\psi(x^{\delta-\alpha'}) > \psi(x^{\delta-\alpha})$. So we can define the function $\pi_{\mathcal{J}}: \mathcal{T} \rightarrow \mathcal{T} \cup \{0\}$ given by $\pi_{\mathcal{J}}(x^\beta) = x^{\beta-\alpha}$ if $x^\beta \in \text{cone}(x^\alpha)$ and $\pi_{\mathcal{J}}(x^\beta) = 0$ if $x^\beta \in \mathbf{N}(J)$.

Moreover, \mathcal{J} is also noetherian; indeed we can define an ordering function $\varphi: \mathcal{T} \rightarrow W \cup \{\perp\}$ (where \perp is a new element and $e > \perp$ for every $e \in W$) given by $\varphi(x^\delta) = \psi(\pi_{\mathcal{J}}(x^\delta))$ if $x^\delta \in J$ and $\varphi(x^\delta) = \perp$ if $x^\delta \in \mathbf{N}(J)$. The condition that φ must satisfy comes from condition StOr4, since $\psi(x^\eta) = \varphi(x^{\eta+\alpha})$ and $\psi(x^{\eta+\gamma-\alpha'}) = \varphi(x^{\eta+\gamma})$. Thus, by Proposition 7.2, \mathcal{J} is also confluent.

We say that g has a $\psi(x^\gamma)$ – SLRep if it has a $\varphi(x^{\gamma+\alpha})$ – SLRep where x^α is an element of M s.t. $x^\gamma \in \tau_\alpha$ (notice that such a term always exists; for example there is an $x^\alpha \in M$ s.t. $\tau_\alpha = \mathcal{T}$).

Lemma 12.3. Let \mathcal{F} be a marked set over a stably ordered RS \mathcal{J} and let $x^\alpha, x^{\alpha'}$ be two distinct elements of M . Suppose that $x^{\eta'+\alpha'} = x^{\eta+\alpha} \in \text{cone}(x^{\alpha'})$, then for each term x^δ in $\text{Supp}(x^\eta f_\alpha) \cap J$ and in $\text{Supp}(x^{\eta'} f_{\alpha'}) \cap J$ it holds $\varphi(x^\delta) < \psi(x^\eta)$.

In particular $x^\eta f_\alpha - x^{\eta'} f_{\alpha'} \xrightarrow{\mathcal{F}\mathcal{J}}_\star 0$ iff $x^\eta f_\alpha - x^{\eta'} f_{\alpha'}$ has a $\psi(x^\eta)$ – SLRep.

Proof. We examine three possible cases.

For $x^\delta = x^{\eta'+\alpha'}$, we get $\varphi(x^{\eta'+\alpha'}) = \psi(x^{\eta'}) < \psi(x^\eta)$ by condition StOr3.

If $x^\delta \in \text{Supp}(x^{\eta'} f_{\alpha'} - x^{\eta'+\alpha'})$, i.e. $x^\delta = x^{\eta'+\gamma}$ with $x^\gamma \in \lambda_{\alpha'}$, then $\varphi(x^{\eta'+\gamma}) < \varphi(x^{\eta'+\alpha'})$ since \mathcal{J} is ordered and $\varphi(x^{\eta'+\alpha'}) < \psi(x^\eta)$ as already proved.

If $x^\delta \in \text{Supp}(x^\eta f_\alpha - x^{\eta+\alpha})$, i.e. $x^\delta = x^{\eta+\gamma}$ with $x^\gamma \in \lambda_\alpha$, then $\varphi(x^\delta) < \varphi(x^{\eta+\alpha})$ by condition StOr4 so we can conclude using the first case. \square

Theorem 12.4. *Let \mathcal{F} be a marked set over a stably ordered RS \mathcal{J} . Then, the property for \mathcal{F} of being a marked basis is equivalent to*

$$(5) \quad \forall x^\beta \in M, \forall x^\varepsilon \text{ minimal in } \mathcal{T} \setminus \tau_\beta \text{ w.r.t. the divisibility, it holds } x^\varepsilon f_\beta \xrightarrow{\mathcal{F}\mathcal{J}}_\star 0.$$

If moreover \mathcal{J} has multiplicative variables, then it is also equivalent to the previous ones:

$$(6) \quad \forall x^\beta \in M, \forall x_i \notin \tau_\beta \text{ it holds } x_i f_\beta \xrightarrow{\mathcal{F}\mathcal{J}}_\star 0.$$

Proof. Due to Theorem 8.2 it is clear that for a marked basis, (5) and (6) hold. So we only prove the non-obvious implications. Suppose that (\mathcal{F}) is not contained in $\langle \tau\mathcal{F} \rangle$. Then, the following set is nonempty

$$U := \{x^\gamma \mid \exists f_\beta \in \mathcal{F} \text{ s.t. } x^\gamma f_\beta \notin \langle \tau\mathcal{F} \rangle\}.$$

Since $(W, >)$ is well founded, the set

$$\psi(U) := \{\psi(x^\gamma) \mid x^\gamma \in U\}$$

has at least a minimal element: suppose that such a minimal element corresponds to x^γ and that $x^\gamma f_\beta \notin \langle \tau\mathcal{F} \rangle$. So $x^\gamma \notin \tau_\beta$; moreover, by StOrd2 and StOrd3, no proper divisors of x^γ are in U .

If x^ε is a divisor of x^γ , minimal in $\mathcal{T} \setminus \tau_\beta$, then by hypothesis $x^\varepsilon f_\beta \xrightarrow{\mathcal{F}\mathcal{J}}_\star 0$. So, by Lemma 12.3, $x^\varepsilon f_\beta$ has a $\psi(x^\varepsilon)$ - SLRep. Multiplying by $x^{\gamma-\varepsilon}$ the representation we get a rewriting of $x^\gamma f_\beta$ as linear combination with coefficients in A of polynomials of the form $x^{\gamma-\varepsilon+\eta_i} f_{\alpha_i}$ with $\psi(x^{\eta_i}) < \psi(x^\varepsilon)$. Thus by StOrd3 we will also have $\psi(x^{\gamma-\varepsilon+\eta_i}) < \psi(x^\gamma)$. By minimality of $\psi(x^\gamma)$ in $\psi(U)$, all the summands $x^{\gamma-\varepsilon+\eta_i} f_{\alpha_i}$ belong to $\langle \tau\mathcal{F} \rangle$ and so does also $x^\gamma f_\beta$, in contradiction with the previous hypotheses.

The second statement directly follows from the first; in fact $\{x_i \notin \tau_\beta\}$ is a minimal basis of $\mathcal{T} \setminus \tau_\beta$. \square

Remark 12.5. Consider a polynomial $x^\varepsilon f_\beta$ as stated in Theorem 12.4 and suppose that $x^{\varepsilon+\beta} \in \text{cone}(x^\alpha)$. Then $S(f_\alpha, f_\beta)$ coincides with $x^\varepsilon f_\beta - x^{\varepsilon+\beta-\alpha} f_\alpha$. Indeed by minimality of x^ε in $\mathcal{T} \setminus \tau_\beta$ each proper divisor x^δ of x^ε belongs to τ_β so it cannot also belong to $\text{cone}(x^\alpha)$.

Anyway, the condition concerning the S-polynomials is not sufficient to ensure the minimality of x^γ in $\mathcal{T} \setminus \tau_\beta$. In other words, the conditions required in Theorem 12.4 are weaker than the ones of Corollary 10.3.

13. SPECIALIZATIONS 1: BASES CONSISTENT WITH A TERM ORDER

In this section we examine the most important types of known polynomial bases consistent with a term order, showing that they can be redefined in our language. In this section \prec will always stand for a term order in \mathcal{T} .

13.1. Gröbner type RSs. Instinctively, we associate to Gröbner bases the idea of consistency with a term order, i.e. the fact that the head terms are bigger than any term in the tails w.r.t. a fixed term order.

Anyway, there is another important feature characterizing them, and we can express it in our language, saying that *Gröbner bases are RSs with maximal cones*.

In this paper, the thing we are more interested in, is the reduction procedure $\xrightarrow{\mathcal{F}\mathcal{J}}$, associated to a marked set \mathcal{F} , rather than the marked set (or basis) itself. The reduction

depends both on \mathcal{F} and on the RS \mathcal{J} , and in particular by the set of multiplicative terms.

Consider a monomial ideal J , a set of generators M and the RS $\mathcal{J} = (M, \lambda, \tau)$ defined as follows

TABLE 1. Gröbner bases

M	A set of generators of a monomial ideal J	The minimal monomial basis of a monomial ideal J
λ	$\lambda_\alpha = \{x^\gamma \in \mathcal{T} \text{ s.t. } x^\gamma \prec x^\alpha\}$	$\lambda_\alpha = \{x^\gamma \in \mathcal{T} \text{ s.t. } x^\gamma \prec x^\alpha\} \cap \mathbf{N}(J)$
τ	$\tau_\alpha = \mathcal{T}$	$\tau_\alpha = \mathcal{T}$

These are RSs consistent with a term order and so also noetherian with ordering function $Id_{\mathcal{T}}: \mathcal{T} \rightarrow (\mathcal{T}, \succ)$. The marked bases over the variation of \mathcal{J} presented in the third column are all and the only *reduced* Gröbner bases.

In this context, Reeves-Sturmfels Theorem (Theorem 5.9) says that a RS with maximal cones is noetherian iff it is consistent with a term order.

Thus Gröbner bases relative to \succ with initial ideal J are all and only the marked bases over a RS \mathcal{J} presented in the second column of the table.

Since the cones are maximal, a marked set \mathcal{F} is a basis iff $\xrightarrow{\mathcal{F}\mathcal{J}}$ is confluent.

Following Buchberger's algorithm, the test can be performed via the reduction of a limited number of S-polynomials among elements of \mathcal{F} .

Indeed, the Main Theorem of Gröbner bases Theory [15, 2.2] declares that a generating set \mathcal{F} is a Groebner basis if and only if each S-polynomial between two elements of \mathcal{F} reduces to 0; the two Buchberger's criteria allow to reduce the number of S-polynomials to be considered.

13.2. Gebauer-Moeller and Staggered linear bases. Gebauer and Moeller [25] present an improving to Buchberger algorithm, trying to avoid some reductions characterizing Gröbner bases.

Definition 13.1. Given the K -vector space $V = K[x_1, \dots, x_n]$, its basis \mathcal{T} and a term order \succ , $B \subset V$ is a *Gauss generating set* of V if $\mathcal{T} = \{\text{LM}_\succ(f), f \in B\}$. It is a *Gauss basis* of V if $\forall x^\delta \in \mathcal{T}, \exists! f \in B, \text{LM}_\succ(f) = x^\delta$.

Definition 13.2. Consider a term order \succ and an ideal I of $K[x_1, \dots, x_n]$. A *staggered linear basis* B of I is given by

- $\{f_1, \dots, f_s\}$ a finite set of generators of I ;
- $\forall i$ a semigroup ideal \mathcal{T}_i s.t. $\bigcup_{i=1}^s \{x^\delta f_i, x^\delta \in \mathcal{T} \setminus \mathcal{T}_i\}$ is a Gauss basis of I .

We can give the following interpretation to Staggered linear bases

Given a generating set $M = \{x^{\alpha_1}, \dots, x^{\alpha_s}\}$ of a monomial ideal J , consider the Gröbner type RS $\mathcal{J} = (M, \lambda, \tau)$ with maximal cones and consistent with the term order

\succ . Consider then a substructure $\mathcal{J}' = (M, \lambda, \tau')$ setting $\tau'_{\alpha_i} = \mathcal{T} \setminus T_i$ where M and the T_i form a Staggered linear basis of J .

The Staggered linear bases are all and the only marked bases over such a substructure \mathcal{J}' . A marked set $\mathcal{F} = \{f_{\alpha_1}, \dots, f_{\alpha_s}\}$ turns out to be a marked basis iff the S -polynomials of the form $S(f_{\alpha_i}, f_{\alpha_j}) = x^{\eta_i} f_{\alpha_i} - x^{\eta_j} f_{\alpha_j}$, s.t. $x^{\eta_i + \alpha_i} \in \text{cone}(x^{\alpha_j})$ reduce to 0.

Gebauer and Moeller [25] gave an algorithm which, given a generating set, extended it to a Staggered linear basis, by iteratively computing the reductions $S(f_{\alpha_i}, f_{\alpha_j}) \rightarrow l$ of each such S -polynomials and, if $l \neq 0$, extending the basis including l and properly adapting the sets T_i in order that the enlarged basis is still Staggered. Unfortunately, it has been proved that this procedure does not terminate.

13.3. Reduction structures of the form defined by Janet.

Definition 13.3. (Janet, 1920) [38, pp .75-9] Given a generating set M of a monomial ideal J and one of its elements x^α , a variable x_j is called *Janet-multiplicative* for x^α w.r.t. M if in M there are no elements x^β s.t. $\deg_i(x^\alpha) = \deg_i(x^\beta)$ for each $i > j$ and $\deg_j(x^\alpha) < \deg_j(x^\beta)$

Moreover, M is *Janet-complete* if each term $x^\gamma \in (M)$ has one and only one decomposition in the product of a term x^α in M by a term x^δ given by the product of Janet-multiplicative variables by x^α .

Janet bases are the marked bases over RSs of the following form:

TABLE 2. Janet bases

M	a J-complete generating set of J
λ	$\lambda_\alpha = \{x^\gamma \in \mathcal{T} \text{ s.t. } x^\gamma \prec x^\alpha\} \cap \mathbf{N}(J)_{ \alpha }$
τ	$\tau_\alpha = \mathcal{T}[\mu_\alpha]$ where μ_α is the set of Janet-multiplicative variables for x^α

These are homogeneous RSs, which are consistent with a term order, they have multiplicative variables and disjoint cones. The RSs of the form defined by Janet need to be consistent with a term order, in order to satisfy noetherianity. Note that the set $M = \{xy\}$ is clearly Janet-complete for $K[x, y]$, but the RS $\mathcal{J} = (M, \lambda_{xy} = \{x^2, y^2\}, \tau_{xy} = \mathcal{T})$ is neither weakly noetherian nor noetherian.

13.4. Janet-like reduction structures.

Gertd and Blinkov [28, 29] define Janet-like bases, similar to Janet bases but more efficient to be computed.

Definition 13.4. Given a nonempty finite set $M \subseteq \mathcal{T}$; $\forall x^\alpha \in M, \forall i, 1 \leq i \leq n$ we will say that x_i^k is a *nonmultiplicative power* for x^α if k is positive and it is the minimum of the differences $\deg_i(x^\beta) - \deg_i(x^\alpha)$ with $x^\beta \in M$, s.t. $\deg_j(x^\beta) = \deg_j(x^\alpha)$ if $j > i$ and $\deg_i(x^\beta) > \deg_i(x^\alpha)$.

Denote the set of all nonmultiplicative powers of x^α in M as $\text{NMP}(x^\alpha, M)$.

The terms of the semigroup ideal

$$\mathcal{NM}(x^\alpha, M) := \{x^\eta \in \mathcal{T}, \exists x^\gamma \in \text{NMP}(x^\alpha, M), x^\gamma \mid x^\eta\}$$

are called *J-nonmultipliers* of $x^\alpha \in M$.

The elements of $\mathcal{M}(x^\alpha, M) := \mathcal{T} \setminus \mathcal{NM}(x^\alpha, M)$ (which are an order ideal) are called *J-multipliers* of $x^\alpha \in M$.

Elements $x^\alpha \in M$ are called *J-divisors* of $x^\beta \in \mathcal{T}$ if $x^\beta = x^\alpha x^\gamma$ with x^γ J-multiplicative for x^α .

The set M is called *J-complete* if each term of (M) is the product of an element of M by one of its *J-multipliers*.

A marked set $F \subset \mathcal{P}$ on $\mathcal{J} = (M, \lambda, \tau)$ is called *J-autoreduced* if $x^\alpha \neq x^{\alpha'}$ for each $x^\alpha, x^{\alpha'} \in M$ and $\lambda_\alpha \subset \mathbf{N}(J)$ for each $x^\alpha \in M$.

Notice that, in our notation, the variables ordering is reversed w.r.t. [28, 29].

TABLE 3. Janet-like bases

M	a J-complete generating set of J
λ	$\lambda_\alpha = \{x^\beta \in \mathcal{T}, x^\beta \prec x^\alpha\} \cap \mathbf{N}(J)$
τ	$\tau_\alpha = \{x^\gamma \text{ J-multiplier for } x^\alpha\}$

In [28], Gerdt and Blinkov show that a term cannot have two different Janet-like divisors and that a Janet divisor is also Janet-like but the converse does not hold in general.

Given $U \subset \mathcal{T}$ there is always a minimal *J-completion*, s.t. it is a subset of any other *J-completion* of U . In [29], fixed a term order \prec , the reduction procedure and the notion of Janet-like basis are defined.

Similarly to the Gröbner case, we have noetherianity given by consistency with a term order. In [29], the following criterion for Janet-like bases is proved.

Theorem 13.5 (Gerdt-Blinkov). *A J-autoreduced set $F \subset \mathcal{P}$ is a Janet-like basis of $I = (F)$ iff $\forall f \in F, \forall p \in \text{NMP}(\mathbf{T}(f), \mathbf{T}\{F\} := \{\mathbf{T}(g), g \in F\}) : pf \rightarrow_\star 0$.*

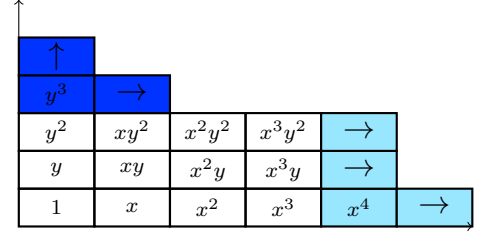
Definition 13.6. Given a set M , a *J-complete* set \overline{M} is called *J-completion* of M if $M \subset \overline{M}$ and $C_J(\overline{M}) = C(M)$ where $C_J(M) := \{x^{\alpha+\gamma}, x^\alpha \in M, x^\gamma \in \mathcal{M}(x^\alpha, M)\}$ and $C(M) := \{x^{\alpha+\eta}, x^\alpha \in M, x^\eta \in \mathcal{T}\}$.

Example 13.7. If in some cases as $M = \{x^3, xy, y^2\} \subseteq K[x, y]$ this construction produces a Janet complete system (obviously also *J-complete*) it does not hold for the following

one.

Fixed $M = \{x^4, y^3\} \subseteq \mathbf{k}[x, y]$, it holds $\mathcal{NM}(x^4, M) = \{\tau y^3, \tau \in \mathcal{T}\}$, $\mathcal{NM}(y^3, M) = \emptyset$ so $\mathcal{M}(x^4, M) = \{x^h, h \geq 0\} \cup \{x^k y^l, k \geq 0\} \cup \{x^l y^2, l \geq 0\}$, whereas $\mathcal{M}(y^3, M) = \mathcal{T}$, as visualized in the picture.

Even if the nonmultiplicative powers are actually powers of nonmultiplicative variables according to Janet [38], the obtained system is J -complete but it is different from a Janet complete system. On the other hand, M is not complete according to Janet.



14. SPECIALIZATIONS 2: STABLY ORDERED BASES

14.1. Reduction structures of Pommaret form.

Definition 14.1 (Janet). A set of terms M is called *stably complete* if it is Janet-complete and for each term $x^\alpha \in M$ its Janet-multiplicative variables are exactly the ones smaller or equal than the minimal appearing in x^α .

The monomial ideals having a stably complete generating set are the quasi stable ones.

Definition 14.2. A monomial ideal J is called *quasi stable* if it holds

$$\tau \in J, x_j > \min(\tau) \implies \exists t \geq 0 : \frac{x_j^t \tau}{\min(\tau)} \in J.$$

Consider a quasi stable monomial ideal J and a term order \succ . Pommaret bases are the marked bases on a RS \mathcal{J} of the type defined in the central column of the following table

TABLE 4. Pommaret

	Coherent with \prec	Term order free
M	A Pommaret basis of J	A Pommaret basis of J
λ	$\lambda_\alpha = \{x^\gamma \in \mathcal{T} \text{ s.t. } x^\gamma \prec x^\alpha\} \cap \mathbf{N}(J)_{ \alpha }$	$\lambda_\alpha = \mathbf{N}(J)_{ \alpha }$
τ	$\tau_\alpha = \mathcal{T}[\mu_\alpha]$ where $\mu_\alpha = \{x_i \leq \min(x^\alpha)\}$	$\tau_\alpha = \mathcal{T}[\mu_\alpha]$ where $\mu_\alpha = \{x_i \leq \min(x^\alpha)\}$

This \mathcal{J} is a homogeneous RS with disjoint cones and it is stably ordered setting $W = \mathcal{T}$, $\psi = Id$, and $>$ the lexicographic term order (see Definition 12.1).

For these RSs the criterion for marked bases is given by Theorem 12.4.

Even if the marked bases over a RS of Pommaret type are Gröbner bases, we cannot consider such a RS as particular case of a RS of Gröbner type, since the reduction procedure $\xrightarrow{\mathcal{FJ}}$ is different from the one of the Gröbner case, which has maximal cones.

A difference influencing the efficiency of the reduction procedure and which is the reason leading to such a definition, concerns the criterion characterizing marked bases, given by Theorem 12.4, that is “linear”. Notice that not all monomial ideals J are quasi stable so they do not necessarily have a finite stably complete generating set.

For more details on Pommaret bases see [51, IV.55.5], [54, 53].

The good properties of Pommaret bases, make us consider less important the fact that the terms in λ_α are chosen so that they are smaller than x^α w.r.t. a fixed term order. In [17], [6] and [16] (see these papers for further details) marked sets and bases of Pommaret type are introduced, but they differ from the usual Pommaret bases because they are term ordering free, i.e. associated to a RS of the form presented in the last column of the previous table.

In [16] the set of terms M is called star-set of the ideal J , supposed quasi-stable. \mathcal{J} has disjoint cones [16, Lemma 3.17], and it is stably ordered [16, Lemma 3.20]. The linearity of the conditions characterizing marked bases, i.e. the analogous of Theorem 12.4 is proved in the strongly stable case in [6] and generalized for stable ideals in [16, Theorem 5.13].

14.2. The zero-dimensional case and border bases. We examine in depth the zero-dimensional case, since it is suitable for many observations.

Let J be a monomial ideal in $A[x_1, \dots, x_n]$ s.t. $N(J)$ is a finite set. An important concept in many papers on this case is the one of *border*.

Definition 14.3. The border of J (or of $N(J)$) is the set of terms

$$B(J) := x_1 N(J) \cup \dots \cup x_n N(J) \setminus N(J).$$

Clearly $B(J)$ contains the monomial basis of J , but in general as a proper subset.

We can characterize the elements of the border as follows:

$$x^\eta \in B(J) \iff \exists x_j : x^\eta / x_j \in N(J).$$

It follows then that the divisors of an element in the border are all contained in $N(J) \cup B(J)$. In many constructions of marked bases over the border, one considers a fixed term order and supposes that in each marked polynomial the elements in the tails are smaller than the head w.r.t. such a term order; anyway there also exist some bases, marked on $B(J)$ without this constraint (see [41] and [1]).

Note that the notion of border bases was originally introduced in [48, 49], but in a context with no connection with reduction structures (actually being a reduction-free approach for computing canonical forms).

Our construction is not compatible with Mourrain improved formulation of border bases in [52] under the notion of *connected to 1*; indeed there it is not required the head terms to be a semigroup ideal nor the escalier to be an order ideal.

We can give a reformulation of these definitions in our language, defining a RS \mathcal{J} as follows. Let $M = \{x^{\alpha_1}, \dots, x^{\alpha_s}\}$ be a list, formed by the elements of $B(J)$, ordered in an arbitrary way. Then, we associate to each term x^γ in J the last term x^{α_i} of the M dividing x^γ .

Notice that this is actually a RS and that the cones are disjoint, as proved in

Lemma 14.4. *Under the previous hypotheses (and w.r.t. the previous notation)*

i) *for each $x^{\alpha_i} \in M$ the set $\tau_{\alpha_i} = \{x^\eta \in \mathcal{T} \text{ s.t. } \forall j > i : x^{\alpha_j} \nmid x^{\eta + \alpha_i}\}$ is an order ideal.*

TABLE 5. Border bases

M	The border basis $B(J) = \{x^{\alpha_1}, \dots, x^{\alpha_s}\}$
λ	$\lambda_{\alpha_i} = N(J)$
τ	$\tau_{\alpha_i} = \{x^\eta \in \mathcal{T} \text{ s.t. } \forall j > i : x^{\alpha_j} \nmid x^{\eta+\alpha_i}\}$

ii) Setting $\text{cone}(x^{\alpha_i}) = x^{\alpha_i}\tau_{\alpha_i}$, it holds $\bigcup_{x^{\alpha_i} \in M} \text{cone}(x^{\alpha_i}) = J$.

Proof. i) Let $x^\eta \in \tau_\alpha$ and $x^\varepsilon | x^\eta$. If some $x^{\alpha'}$ subsequent to x^α in the list divides $x^{\varepsilon+\alpha}$ then it divides also $x^{\eta+\alpha}$ and this contradicts $x^\eta \in \tau_\alpha$.

ii) We prove that for each $x^\beta \in J$ there is a term $x^{\alpha_i} \in B(J)$ s.t. $x^{\beta-\alpha_i} \in \tau_{\alpha_i}$. This trivially follows by construction and from the fact that $B(J)$ is a generating set for J . \square

In [41] the authors consider (in the case $A = K$ field) a reduction process w.r.t. a marked set \mathcal{F} over \mathcal{J} (called *border pre-basis*). Roughly speaking, the terms in $B(J)$ are ordered in increasing order by degree and a term x^γ in J is reduced by the element in \mathcal{F} whose head has maximal degree among the terms of $B(J)$ dividing x^γ . In order to prove the noetherianity of this reduction process, a function $\text{ind}_{B(J)} : J \cap \mathcal{T} \rightarrow \mathbb{N}$ called *index* is defined, associating to each term x^γ in J the minimum degree of its divisors x^δ such that $x^{\gamma-\delta} \in B(J)$. In some sense the function $\text{ind}_{B(J)}$ plays the role of a sort of ordering function with values in the well founded set $(\mathbb{N}, >)$. More precisely, $\text{ind}_{B(J)}$ shares some properties with the function ψ that characterizes the stably reduced RSs. Anyway, we cannot exactly repeat this construction, since in general the index does not satisfy the condition StOr3 in Definition 12.1.

Notice that in [41] there are no characterizations of marked bases using the reduction procedure; the presented characterization is based, as for Mourrain's work, on the commutativity of multiplication matrices.

If we modify a bit this idea, we can get a stably ordered RS. We only have to reorder the elements of $M = B(J)$ in increasing order w.r.t. a term order \succ (not necessarily degree compatible). Now Theorem 12.4 gives a finite set of reductions to control in order to decide whether \mathcal{F} is a marked basis.

Lemma 14.5. *Under the previous hypotheses (and w.r.t. the previous notation), let \succ be a term order. If the terms in $M = B(J)$ are ordered in such a way that $x^{\alpha_i} \prec x^{\alpha_{i+1}}$, then \mathcal{J} is stably ordered with $\psi = \text{Id}_{\mathcal{T}} : \mathcal{T} \rightarrow (\mathcal{T}, \succ)$.*

Proof. The function ψ trivially verifies StOr1 and StOr2 of Definition 12.1 since \succ is a term order.

For StOr3, let $x^{\eta+\alpha} = x^{\eta'+\alpha'} \in \text{cone}(x^{\alpha'})$ with $x^\alpha \neq x^{\alpha'}$. According to our construction, it holds $x^{\alpha'} \succ x^\alpha$, hence $x^{\eta'} \prec x^\eta$.

For StOr4, let $x^\gamma \in \lambda_\alpha$, $x^\eta \in \tau_\alpha$ s.t. $x^{\eta+\gamma} \in \text{cone}(x^{\alpha'})$. Then, $x^\gamma \notin J$, whereas $x^{\eta+\gamma} \in J$, so that we can find a term $x^\varepsilon \neq 1$ s.t. $x^{\alpha''} = x^{\varepsilon+\gamma}$ belongs to $B(J)$ and divides $x^{\eta+\gamma}$. According to our definition of $\text{cone}(x^{\alpha'})$, we have $x^{\alpha''} \preceq x^{\alpha'}$. Thus $x^\eta \succ x^{\eta-\varepsilon} = x^{\eta+\gamma-\alpha''} \succeq x^{\eta+\gamma-\alpha'}$. \square

We now show in some examples the consequences of modifying the order of the elements of M .

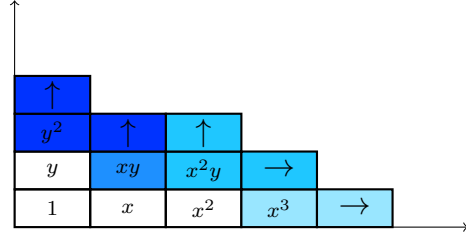
Example 14.6. Let $J = (x^3, x^2y^2, y^3) \subset A[x, y]$. Its border can be written (in increasing order by degree) as $x^3, y^3, x^2y^2, x^3y, xy^3$. The multiplicative sets of the corresponding RS \mathcal{J} are: $\tau_{x^3} = \mathcal{T}[x]$, $\tau_{y^3} = \mathcal{T}[y]$, $\tau_{x^2y^2} = \{1\}$, $\tau_{x^3y} = \mathcal{T}[x, y]$, and $\tau_{xy^3} = \{x^ay^b, a \geq 0, 0 \leq b \leq 1\}$. Notice that \mathcal{J} is not a RS with multiplicative variables.

According to the criterion presented in Theorem 12.4 in order to know whether a marked set $\mathcal{F} = \{f_{x^3}, f_{y^3}, f_{x^2y^2}, f_{x^3y}, f_{xy^3}\}$ is a marked basis we would control the reduction of the following polynomials $yf_{x^3}, xf_{y^3}, xf_{x^2y^2}, yf_{x^2y^2}, y^2f_{x^3y}$.

Now we reorder the terms in $B(J)$ in increasing order w.r.t. DegLex (induced by $x \prec y$) getting $x^3, y^3, x^3y, x^2y^2, xy^3$. In this case the multiplicative sets are $\tau'_{x^3} = \tau'_{x^3y} = \tau'_{x^2y^2} = \mathcal{T}[x]$, $\tau'_{y^3} = \mathcal{T}[y]$ and $\tau'_{xy^3} = \mathcal{T}[x, y]$. We get a stably ordered RS \mathcal{J}' with multiplicative variables. Following Theorem 12.4 to decide whether \mathcal{F} is a marked basis, we only have to check the reduction of $yf_{x^3}, xf_{y^3}, yf_{x^3y}, yf_{x^2y^2}$, all of them of “linear type”.

Anyway, we cannot generalize what observed in the previous example, since re-ordering the terms of the border w.r.t. DegLex (or a degree compatible term order) we do not obtain necessarily a RS with multiplicative variables.

Example 14.7. Ordering the border of $J = (x^3, xy, y^2) \subset A[x, y]$, w.r.t DegLex ($x \prec y$) we obtain xy, y^2, x^3, x^2y with cones $\tau_{xy} = \{1\}$, $\tau_{y^2} = \{x^ly^h, l \leq 1, h \geq 0\}$, $\tau_{x^3} = \mathcal{T}[x]$, $\tau_{x^2y} = \mathcal{T}[x, y]$, as in the picture. The term x^2 is one of the minimal elements of $\mathcal{T} \setminus \tau_{y^2}$ w.r.t. divisibility. This means that in order to verify that a marked set \mathcal{F} is also a marked basis we have also to reduce $x^2f_{y^2}$, which is not of “linear type”.



The most convenient choice in general is to forget the degree and reorder the terms w.r.t. Lex.

Theorem 14.8. Let J be a zero-dimensional monomial ideal and let $M = B(J)$ be its border. Consider M ordered w.r.t. the lexicographic term order \prec_{Lex} and let \mathcal{J} be the associated RS according to Table 5.

Then \mathcal{J} has multiplicative variables, which for every $x^\alpha \in B(J)$ coincide with the Janet-multiplicative variables for x^α w.r.t. $B(J)$, so $B(J)$ is a Janet complete system.

Proof. Let μ_α the set of Janet-multiplicative variables for $x^\alpha \in B(J)$. We have to prove that $\tau_\alpha = \mathcal{T}[\mu_\alpha]$.

\supseteq Consider $x^\eta \in \mathcal{T}[\mu_\alpha]$ and verify that $x^\eta \in \tau_\alpha$ i.e. that there is no term $x^\beta \in M$ dividing $x^{\eta+\alpha}$ and s.t. $x^\beta \succ_{\text{Lex}} x^\alpha$.

Suppose by contradiction that such a term x^β exists. Let x_j be s.t. $\deg_i(x^\beta) = \deg_i(x^\alpha)$ for each $i > j$ and $\deg_j(x^\beta) > \deg_j(x^\alpha)$. By definition of Janet-multiplicative variable, $x_j \notin \mu_\alpha$. We then get a contradiction, since by $\deg_j(x^{\eta+\alpha}) \geq \deg_j(x^\beta) > \deg_j(x^\alpha)$ follows that $x_j \mid x^\eta$ so, by hypothesis, $x_j \in \mu_\alpha$.

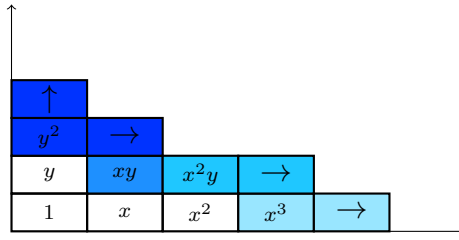
\subseteq It is sufficient to prove that $x_j \notin \mu_\alpha$ implies $x_j \notin \tau_\alpha$.

If $x_j \notin \mu_\alpha$, by the definition of Janet-multiplicative variable there is a term $x^\beta \in B(J)$ with $\deg_i(x^\beta) = \deg_i(x^\alpha)$ for each $i > j$ and $\deg_j(x^\beta) > \deg_j(x^\alpha)$. We prove then that the border contains also an element $x^{\beta'}$ with $\deg_i(x^{\beta'}) = \deg_i(x^\alpha)$ for each $i > j$ and $\deg_j(x^{\beta'}) = \deg_j(x^\alpha) + 1$, so that $x_j \notin \tau_\alpha$.

Consider the term x^γ obtained by x^α evaluating at 1 the variables x_i , $i < j$. By construction $x_j x^\gamma \mid x^\beta$ which is in the border; thus $x_j x^\gamma \in B(J) \cup N(J)$. Moreover $x_j x^\gamma$ also divides $x_j x^\alpha$, which belongs to J . Then, we find the wanted term $x^{\beta'} \in B(J)$ in the set of the multiples of $x_j x^\gamma$ dividing $x_j x^\alpha$. \square

Example 14.9. Consider again the monomial ideal of Example 14.7. The border of $J = (x^3, xy, y^2)$, ordered w.r.t. Lex is x^3, xy, x^2y, y^2 . The multiplicative sets of the corresponding RS \mathcal{J}'' are $\tau_{x^3}'' = \tau_{x^2y}'' = \mathcal{T}[x]$, $\tau_{xy}'' = \{1\}$, $\tau_{y^2}'' = \mathcal{T}[x, y]$.

Thus, \mathcal{J}'' is a stably ordered RS with multiplicative variables (coinciding with the Janet-multiplicative ones).



To conclude, we present a monomial ideal J for which the border basis (with terms ordered w.r.t. the Lex term order) is not a Pommaret basis, even though J has both type, being quasi stable.

Example 14.10. The terms of the border of $J = (x^3, x^2y, y^3) \subset A[x, y]$ ordered w.r.t. Lex are $x^3, x^2y, x^2y^2, y^3, xy^3$. We get: $\tau_{x^3} = \tau_{x^2y} = \tau_{x^2y^2} = \mathcal{T}[x]$, $\tau_{y^3} = \mathcal{T}[y]$, $\tau_{xy^3} = \mathcal{T}[x, y]$.

The set of controls one has to perform in order to decide whether a marked set $\mathcal{F} = \{f_{x^3}, f_{x^2y}, f_{x^2y^2}, f_{y^3}, f_{xy^3}\}$ involves the reduction of $yf_{x^3}, yf_{x^2y}, yf_{x^2y^2}, xf_{y^3}$.

Notice that the sets of multiplicative variables of y^3 and xy^3 do not coincide with the ones w.r.t. Pommaret. Indeed, in the Pommaret basis $\{x^3, x^2y, x^2y^2, y^3\}$ of J there is one term less than in the border basis. At least in this case, in order to determine all the ideals in $A[x, y]$ whose quotient is a free A -module with basis $N(J)$, it would be more convenient to use the Pommaret basis, instead of the border basis. Indeed, looking at the RS given in Table 4 we find that the set of controls that are needed involves only three reductions: $yf_{x^3}, yf_{x^2y}, yf_{x^2y^2}$.

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